

TROPICAL GLUING FORMULAE FOR GROMOV–WITTEN INVARIANTS

BRETT PARKER

ABSTRACT. We prove two tropical gluing formulae for Gromov–Witten invariants of exploded manifolds, useful for calculating Gromov–Witten invariants of a symplectic manifold using a normal-crossing degeneration. The first formula generalizes the symplectic-sum formula for Gromov–Witten invariants. The second formula is stronger, and also generalizes Kontsevich and Manin’s splitting and genus-reduction axioms. Both tropical gluing formulae have versions incorporating gravitational descendants.

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1. INTRODUCTION

This paper explains the following tropical gluing formulae for Gromov–Witten invariants.

$$(1) \quad \eta|_{\gamma} = \frac{k_{\gamma}}{|\mathrm{Aut} \, \gamma|} i_{!}^{[\gamma]} \Delta^* \prod_v \eta^{[\gamma_v]}$$

$$(2) \quad \mu|_{\gamma} = \frac{1}{|\mathrm{Aut} \, \gamma|} I_{!}^{[\gamma]} \Delta^* \prod_v \mu^{[\gamma_v]}$$

On the left, η and μ represent Gromov–Witten invariants. In the case of a compact symplectic manifold B , η and μ encode the usual Gromov–Witten invariants obtained using evaluation of curves with n marked points to B^n and $B^n \times \bar{M}_{g,n}$ respectively. The notation $|_{\gamma}$ indicates the contribution of a tropical curve γ to these invariants. On the right, $\eta^{[\gamma_v]}$ and $\mu^{[\gamma_v]}$ indicate relative¹ Gromov–Witten invariants associated to vertices v of γ , and the remaining terms are elementary instructions for combining these relative invariants to compute η and μ . The relative invariants themselves are subject to the same gluing formulae, which often suffice to determine them; see for example [18, 26].

Formula (1) generalizes the symplectic-sum formula for Gromov–Witten invariants, [11, 6, 10], to normal-crossing or log-smooth degenerations, and symplectic analogues;² see [16] for an exposition of the simplest generalization — the case of a triple-product. Formula (2) also gives a degeneration formula for Gromov–Witten invariants in these situations, but includes the contribution of cohomology classes from Deligne–Mumford space; without applying any degeneration, it specializes to Kontsevich and Mannin’s splitting and genus-reduction axioms for Gromov–Witten invariants of symplectic manifolds. We also prove a version of each formula incorporating gravitational descendants, the contribution of Chern classes of tautological line-bundles over the moduli stack of curves.

This paper’s action takes place within the category of exploded manifolds;³ our results apply to compact symplectic manifolds using degeneration. We can explode any normal-crossing degeneration of a symplectic manifold B to produce a smooth family of exploded manifolds; smooth fibers of the original degeneration are unchanged, but the singular fiber is replaced with a family of exploded manifolds. Gromov–Witten invariants do not change in families of exploded manifolds, so the Gromov–Witten invariants of B can be computed using any other fiber, and it is convenient to use one of the fibers \mathbf{B} replacing the singular fiber. For examples, see [26, 16, 18].

Let us explain the tropical curves γ appearing in our gluing formulae, (1) and (2). Each exploded manifold \mathbf{B} has a tropical part, $\underline{\mathbf{B}}$, consisting of a complex of integral-affine polytopes. For example, if \mathbf{B} arises as a fiber of a normal-crossing degeneration with singular fiber B' , $\underline{\mathbf{B}}$ will be the dual complex of simplices with a vertex for each component of B' , and a n -simplex for every intersection of $(n+1)$ components. Each curve in \mathbf{B} is a map $f: \mathbf{C} \rightarrow \mathbf{B}$ which itself has a tropical

¹These relative invariants are defined using exploded manifolds, however they are roughly equivalent to Ionel’s relative invariants from [5], and in the algebraic setting are equivalent to log Gromov–Witten invariants, [4, 2, 1].

²For an appropriate symplectic analogue of normal-crossing divisors, see [14], and for a symplectic analogue of a normal-crossing degeneration, see [13].

³See [21] or [17] for an introduction to exploded manifolds. In the algebraic setting, using exploded manifolds is almost equivalent to using log schemes, and equivalent gluing formulae should be provable entirely within the framework of log schemes. For construction of log Gromov–Witten invariants, see [4, 2, 1]. See [22] for a log-exploded dictionary.

part, $\underline{f}: \underline{\mathbf{C}} \rightarrow \underline{\mathbf{B}}$, where $\underline{\mathbf{C}}$ is a graph with a complete integral-affine structure on its edges, and \underline{f} is integral-affine. A tropical curve γ in $\underline{\mathbf{B}}$ is such an integral-affine map from a complete integral-affine graph.

All exploded manifolds \mathbf{B} also have a smooth part, $[\mathbf{B}]$ consisting of a complex of smooth manifolds. In the above case, $[\mathbf{B}]$ is isomorphic to the singular fiber B' , and $[f]: [\mathbf{C}] \rightarrow [\mathbf{B}]$ is a holomorphic curve in B' , with a component for each vertex of $\underline{\mathbf{C}}$, a node for each internal edge of $\underline{\mathbf{C}}$, and a marked point corresponding to each end of $\underline{\mathbf{C}}$ (isomorphic to $[0, \infty)$). Although the reader might intuitively think in terms of these smooth parts, consideration of the extra tropical structure is essential for correct gluing and degeneration formulae.

Gromov–Witten invariants of \mathbf{B} decompose into a sum of (virtual) contributions of tropical curves γ , or rather the holomorphic curves with tropical part isomorphic to γ . Our gluing formulae compute this contribution of γ in terms of relative invariants, $\eta^{[\gamma_v]}$ and $\mu^{[\gamma_v]}$. These relative invariants are Gromov–Witten invariants of an exploded manifold $\check{\mathbf{B}}|_v$ created by completing the stratum of \mathbf{B} containing v using the tropical completion described in section 7 of [25]. In the case that $[\mathbf{B}]$ is the degenerate fiber of a normal-crossing degeneration and v is in a corner of $\underline{\mathbf{B}}$, the corresponding stratum of $[\mathbf{B}]$ is a manifold with a normal-crossing divisor, and $\check{\mathbf{B}}|_v$ is the explosion⁴ of this stratum. Our relative invariants $\eta^{[\gamma_v]}$ and $\mu^{[\gamma_v]}$ are Gromov–Witten invariants of this stratum relative to its normal-crossing divisor. If v is in a k -dimensional stratum of $\underline{\mathbf{B}}$, $\check{\mathbf{B}}|_v$ is a k -complex-dimensional bundle over the corresponding k -fold intersection of components of $[\mathbf{B}]$, related to the ‘rubber components’ or ‘expansions’ that appear in [11, 6, 10]. The Gromov–Witten invariants of such ‘expansions’ are not important for the symplectic-sum formula, but are critical for its correct generalization; for simple examples in the case of a triple-product, see [16].

Let us describe our Gromov–Witten invariant η .

$$\eta := ev_! (q^E \hbar^{2g-2+n})$$

Above, q and \hbar are dummy variables whose exponents record the symplectic energy E , and Euler characteristic, $2g - 2 + n$, of curves, and $ev_!$ indicates pushforward⁵ from the virtual fundamental class⁶ using the evaluation map,

$$ev: \mathcal{M}^{st}(\mathbf{B}) \rightarrow \coprod_n (\text{End } \mathbf{B})^n$$

constructed in section 3. The notation $\mathcal{M}^{st}(\mathbf{B})$ indicates a moduli stack of (not-necessarily holomorphic) stable curves in \mathbf{B} with labeled ends⁷. When $\underline{\mathbf{B}}$ is bounded, $\text{End } \mathbf{B} = \mathbf{B}$, and our evaluation map ev simply records the location of ends/punctures of curves. When \mathbf{B} is the explosion of a manifold with a smooth divisor D , $\text{End } \mathbf{B}$ is the disjoint union of \mathbf{B} with a copy of D for each positive integer ‘contact order’.⁸ The evaluation map at an end/puncture lands in \mathbf{B} or the appropriate copy of D , depending on the contact order at that end. In more general cases,⁹ $\text{End } \mathbf{B}$ records

⁴See [21],[17], or [22] for an exposition of the explosion functor.

⁵See [25], section 5.3, for pushforwards from the virtual fundamental class.

⁶See [23] for the construction of an embedded Kuranishi structure for the moduli stack of holomorphic curves, and [25] for the construction of the virtual fundamental class using this embedded Kuranishi structure.

⁷An end of an (exploded) holomorphic curve \mathbf{C} corresponds to an end, or infinite edge, of its underlying tropical curve $\underline{\mathbf{C}}$. Ends correspond to marked points of the underlying nodal curve $[\mathbf{C}]$.

⁸The tropical part of \mathbf{B} in this case is a half-line, as is the tropical part of each end of a curve. The smooth part of a curve in \mathbf{B} has contact order k with D at an end if the derivative of its tropical part at that end is k .

⁹The evaluation map ev is described precisely in section 3.

the possible positions of the ends of curves in \mathbf{B} , and is the disjoint union of \mathbf{B} with exploded manifolds of real-dimension 2 lower than \mathbf{B} .

The relative invariant $\eta^{[\gamma_v]}$ appearing in our gluing formula (1) is the restriction of the corresponding invariant η of $\mathbf{B}|_v$ to the connected component

$$\text{End}^{[\gamma_v]} \mathbf{B}|_v \subset \coprod_n (\text{End } \mathbf{B}|_v)^n$$

containing the image of curves in $\mathbf{B}|_v$ with tropical part γ_v , where γ_v is a tropical curve in $\mathbf{B}|_v$ with a single vertex, and edges created by infinitely lengthening all edges of γ leaving v . Alternatively, we may define

$$\eta^{[\gamma_v]} := ev_!^{[\gamma_v]}(q^E \hbar^{2g-2+n})$$

where

$$ev^{[\gamma_v]}: \mathcal{M}_{[\gamma_v]}^{st}(\mathbf{B}|_v) \longrightarrow \text{End}^{[\gamma_v]}(\mathbf{B}|_v)$$

is our evaluation map ev restricted to the moduli stack $\mathcal{M}_{[\gamma_v]}^{st}(\mathbf{B}|_v) \subset \mathcal{M}^{st}(\mathbf{B}|_v)$ of curves in $\mathbf{B}|_v$ with ends labelled by the ends of γ_v , and having derivatives¹⁰ at these ends equal to the derivatives of the ends of γ_v .

Suppose that γ has n ends. To write our gluing formula, the relationship between $(\text{End } \mathbf{B})^n$ and $\text{End}^{[\gamma_v]} \mathbf{B}|_v$ is encapsulated in the following maps, explained in section 3.

$$\coprod_v \text{End}^{[\gamma_v]}(\mathbf{B}|_v) \xleftarrow{\Delta} \mathbf{Y} \xrightarrow{i^{[\gamma]}} (\text{End } \mathbf{B})^n|_{\underline{ev}\gamma}$$

Each internal edge of γ corresponds to two ends of $\coprod_v \gamma_v$. The corresponding two connected components of $\coprod_v \text{End}(\mathbf{B}|_v)$ are naturally isomorphic, and the map Δ is the inclusion of the diagonal subset of $\coprod_v \text{End}^{[\gamma_v]}(\mathbf{B}|_v)$ using these isomorphism for each internal edge of γ . The map $i^{[\gamma]}$ is a projection which forgets the information from each of these internal edges. In particular, each of the n ends of γ corresponds to a unique end of $\coprod_v \gamma_v$, and the connected component of $(\coprod_v \text{End}(\mathbf{B}|_v))^n$ recording the position of these n ends is naturally isomorphic to $(\text{End } \mathbf{B})^n|_{\underline{ev}\gamma}$, the tropical completion of $(\text{End } \mathbf{B})^n$ at the image of curves with tropical part γ . Our map $i^{[\gamma]}$ is the projection which forgets the factors recording the position of internal edges, followed by this natural isomorphism.

Our first gluing formula

$$\eta|_\gamma = \frac{k_\gamma}{|\text{Aut } \gamma|} i_!^{[\gamma]} \Delta^* \prod_v \eta^{[\gamma_v]}$$

contains two combinatorial factors we have yet to explain. The constant k_γ is the product of the multiplicities m_e of the internal edges of γ , where each internal edge of γ has derivative an integral-vector equal to m_e times a primitive integral-vector. This factor of k_γ arises because our gluing formula follows from a natural fiber-product diagram involving not the exploded manifold $\text{End } \mathbf{B}$, but a corresponding stack $\mathcal{E}\mathbf{B}$ that is the quotient of $\text{End } \mathbf{B}$ by a trivial group action, \mathbb{Z}_{m_e} on the component corresponding to an edge of multiplicity $m_e > 0$, and the infinite group \mathbf{T} on the component corresponding to an edge of multiplicity 0. Our formula also requires division by the size of the automorphism group,¹¹ $\text{Aut } \gamma$, of the tropical

¹⁰If f is a curve in $\mathbf{B}|_v$, and $\mathbf{B}|_v$ is the explosion of a manifold with normal-crossing divisor D , the derivative of an end of f is an integral-vector whose components correspond to contact orders of $[f]$ with components of D .

¹¹We only use automorphisms of γ that fix ends, because we have labeled ends of curves by working with \mathcal{M}^{st} .

curve γ , because the natural fiber-product diagram we use involves labelling edges of curves by the edges of γ .

We also prove a related gluing formula that includes gravitational descendants. Suppose that W is a tautological vectorbundle over $\mathcal{M}^{st}(\mathbf{B})$, so W is a product of tautological line-bundles corresponding to the ends of curves. Let W_v be the corresponding tautological vectorbundle over $\mathcal{M}_{[\gamma_v]}^{st}(\mathbf{B}|_v)$, remembering that some ends of curves in $\mathcal{M}_{[\gamma_v]}^{st}$ correspond naturally to ends of curves in \mathcal{M}^{st} . Define

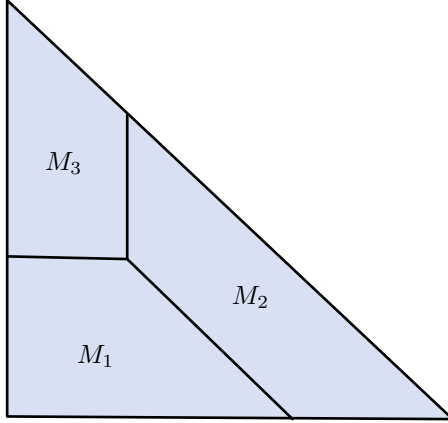
$$\eta(W) := ev_!(q^E h^{2g-2+n} c(W))$$

$$\eta^{[\gamma_v]}(W_v) := ev_!^{[\gamma_v]}(q^E h^{2g-2+n} c(W_v))$$

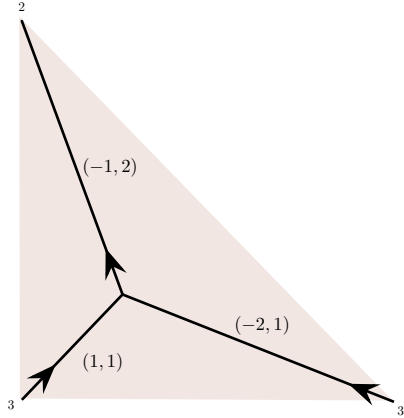
using the pushforward of top Chern-classes $c(W)$ and $c(W_v)$ defined in [25], Remark 5.2. These Gromov–Witten invariants satisfy the following modified version of (1).

$$\eta(W)|_{\check{\gamma}} = \frac{k_{\gamma}}{|\text{Aut } \gamma|} i_!^{[\gamma]} \Delta^* \prod_v \eta^{[\gamma_v]}(W_v)$$

Let us consider the elements of our gluing formula in a quick example, discussed in section 8 of [16]. Degenerate $\mathbb{C}P^2$ into three components M_i isomorphic to CP^2 blown up at 1 point, as pictured in the toric moment-map diagram below.



The picture below is the image of a tropical curve γ in the tropical part $\underline{\mathbf{B}}$ of an exploded manifold \mathbf{B} with smooth part fiber pictured above. The little number at a vertex in the corner denotes the number of ends of γ attached to that vertex, because all these ends are sent to a point in $\underline{\mathbf{B}}$.



Let us consider $\eta^{[\gamma_v]}$ for the above tropical curve. For v_0 the central vertex, $\mathbf{B}|_{v_0}$ is an exploded manifold \mathbf{T}^2 , analogous to $(\mathbb{C}^*)^2$. The relevant components of

$\text{End}(\mathbf{T}^2)$ for our evaluation map are the quotient of \mathbf{T}^2 by the \mathbf{T} -actions of weight $(-1, -1)$, $(2, -1)$, and $(-1, 2)$ respectively, each isomorphic to \mathbf{T} . For this vertex, $\eta^{[\gamma_{v_0}]} = \hbar\theta_0$, where θ_0 is the Poincare dual to the product of the three quotient maps $\mathbf{T}^2 \rightarrow \mathbf{T}^3$.

For v_i the remaining vertices, $\eta^{[\gamma_{v_i}]}$ is a Gromov–Witten invariant of curves in the explosion of M_i relative to $M_i \cap M_j$ and $M_i \cap M_k$. Curves with tropical part γ_{v_i} correspond to curves in M_i with one special point sent to $M_1 \cap M_2 \cap M_3$, having contact order 1 with each divisor, and two or three other special points not contacting any divisor. Evaluation at these points is into \mathbf{T} , and $\text{Expl } M_i$ respectively. Our remaining $\eta^{[\gamma_{v_i}]}$ are as follows:

- $\eta^{[\gamma_{v_3}]} = \hbar q^{2E_{31}+E_{32}}$ in the 0-dimensional cohomology of $(\text{Expl } M_3)^2 \times \mathbf{T}$, where E_{ij} is the symplectic area of $M_i \cap M_j$.
- $\eta^{[\gamma_{v_1}]} = \hbar^2 q^{E_{13}+2E_{12}} \theta_1$, where $\theta_1 \in {}^r H^2((\text{Expl } M_1)^3 \times \mathbf{T})$ has integral 1 over the homology class represented by \mathbf{T} .¹²
- Similar to the case of v_1 , $\eta^{[\gamma_{v_2}]} = \hbar^2 q^{E_{21}+2E_{23}} \theta_2$, where $\theta_2 \in {}^r H^2((\text{Expl } M_2)^3 \times \mathbf{T})$ has integral 1 over the homology class represented by \mathbf{T} .

Our gluing formula uses

$$\begin{array}{c}
 (\text{Expl } M_1)^3 \times (\text{Expl } M_2)^3 \times (\text{Expl } M_3)^2 \\
 \uparrow i^{[\gamma]} \\
 (\text{Expl } M_1)^3 \times (\text{Expl } M_2)^3 \times (\text{Expl } M_3)^2 \times \mathbf{T}^3 \\
 \downarrow \Delta \\
 (\text{Expl } M_1)^3 \times (\text{Expl } M_2)^3 \times (\text{Expl } M_3)^2 \times \mathbf{T}^6
 \end{array}$$

and reads

$$\eta|_{\gamma} = \hbar^6 q^{2(E_{12}+E_{23}+E_{31})} i_1^{[\gamma]} \Delta^*(\theta_1 \wedge \theta_2 \wedge \theta_0) .$$

As both \mathbf{T}^3 and $\theta_1 \wedge \theta_2 \wedge \theta_0$ are 6-dimensional, $\eta|_{\gamma}$ is 0-dimensional, so it suffices to evaluate it at a point. This evaluation amounts to the integral of $\theta'_1 \wedge \theta'_2 \wedge \theta_0$ over \mathbf{T}^3 , where θ'_i is the pullback of the generator of the top-dimensional cohomology on the i th \mathbf{T} -factor of \mathbf{T}^3 . This integral is 3, so $\eta|_{\gamma} = 3\hbar^6 q^{2(E_{12}+E_{23}+E_{31})}$.

In terms of counting curves, this statement translates to there being 3 rigid curves in \mathbf{B} with tropical part γ and the 8 ends constrained to chosen points in M_i . Moreover these curves have genus 0 and symplectic energy $2(E_{12} + E_{23} + E_{31})$, so they are degree 2. These are not all the curves in \mathbf{B} with ends constrained to these points in M_i — there are 9 other rigid curves with different tropical parts, as explained in section 8 of [16].

Let us briefly discuss the Gromov–Witten invariant μ starring in our second gluing formula, (2). This invariant uses an enhanced evaluation map

$$EV: \mathcal{M}^{st}(\mathbf{B}) \rightarrow \mathcal{X}(\mathbf{B}) = \coprod_{g,n} \mathcal{X}_{g,n}(\mathbf{B})$$

discussed in section 5, where for $2g + -2 + n > 0$,

$$\mathcal{X}_{g,n}(\mathbf{B}) = \mathcal{M}_{g,n}^{st}(pt) \times_{\cdot/\mathbf{T}^n} (\mathcal{E} \mathbf{B})^n$$

¹²The notation ${}^r H^*$ indicates we are using refined cohomology, from section 9 of [19].

and $\mathcal{E}\mathbf{B}$ is the natural stack used to prove gluing formula (1), and $\mathcal{M}_{g,n}^{st}(pt)$ is the moduli stack of stable curves mapping to a point.¹³ EV is induced from an evaluation map \mathbf{ev} to $(\mathcal{E}\mathbf{B})^n$ and a stabilization map ev^0 to $\mathcal{M}_{g,n}^{st}$ constructed in section 4.1 of [23]. In the case that \mathbf{B} is bounded $\mathcal{X}_{g,n} = \mathcal{M}_{g,n}^{st}(pt) \times \mathbf{B}^n$ and EV is $ev^0 \times ev$. In the case that \mathbf{B} is the explosion of a manifold with a smooth divisor D , the connected component of $\mathcal{X}_{g,n}$ containing the image of a curve with n_2 points contacting the divisor, and n_1 ordinary points is a \mathbf{T}^{n_2} -bundle over $\mathbf{B}^{n_1} \times \mathbf{D}^{n_2} \times \mathcal{M}_{g,n}$. The precise constructions of EV and $\mathcal{X}_{g,n}$ for all (g, n) appear in section 5.

Our Gromov–Witten invariant μ no longer needs the exponent of \hbar to keep track of Euler class, and is defined

$$\mu := EV_!(q^E)$$

or, in the case of a descendant invariant associated to a complex vectorbundle W ,

$$\mu(W) := EV_!(q^E c(W)) .$$

The relative invariant $\mu^{[\gamma_v]}$ is again the restriction of the corresponding invariant μ for $\mathbf{B}|_v$ to the connected components

$$\mathcal{X}^{[\gamma_v]}(\mathbf{B}|_v) \subset \mathcal{X}(\mathbf{B}|_v)$$

containing the image of curves with tropical part γ_v . Our second gluing formula uses maps

$$\prod_v \mathcal{X}^{[\gamma_v]}(\mathbf{B}|_v) \xleftarrow[\Delta]{} \mathbf{Y}' \xrightarrow{I^{[\gamma]}} \mathcal{X}(\mathbf{B})|_{\underline{EV}\gamma}$$

similar to those that appear in our first gluing formula. It reads

$$\mu|_{\gamma} = \frac{1}{|\mathrm{Aut} \gamma|} I_!^{[\gamma]} \Delta^* \left(\prod_v \mu^{[\gamma_v]} \right)$$

and in the case of descendant invariants using a tautological vectorbundle W on $\mathcal{M}^{st}(\mathbf{B})$ corresponding to tautological vectorbundles W_v on $\mathcal{M}_{[\gamma_v]}^{st}(\mathbf{B}|_v)$, it reads

$$\mu(W)|_{\gamma} = \frac{1}{|\mathrm{Aut} \gamma|} I_!^{[\gamma]} \Delta^* \left(\prod_v \mu^{[\gamma_v]}(W_v) \right) .$$

1.1. Technical assumptions. Throughout, we shall assume that \mathbf{B} is a smooth, complete, exploded manifold with a taming form ω , and a $\bar{\partial}$ -log compatible almost-complex structure J tamed by ω .¹⁴ We shall also assume that the tropical part of \mathbf{B} admits an affine immersion into some \mathbb{R}^n so that [24] establishes the compactness of the moduli stack of holomorphic curves with bounded energy and genus that map to a connected component of $\coprod_n \mathrm{End} \mathbf{B}$. This assumption implies that \mathbf{B} is basic.

Under these assumptions, [20, 23] constructs an embedded Kuranishi structure on the moduli stack of curves in \mathbf{B} . This Kuranishi structure is stronger than many other comparable constructions [3, 12, 7, 15] because it comes naturally embedded in a moduli stack $\mathcal{M}^{st}(\mathbf{B})$ of smooth, stable, not necessarily holomorphic curves. The virtual fundamental class of the moduli stack of holomorphic curves is constructed in [25], where we also see how to integrate differential forms over this virtual class, and push forward differential forms over evaluation maps.

¹³ $\mathcal{M}_{g,n}^{st}(pt)$ is constructed in section 4.1 of [23]. This moduli stack is an exploded orbifold, and the explosion of Deligne-Mumford space relative to its boundary.

¹⁴For definitions of exploded manifolds and concepts such as basic and complete, see [21]. Taming forms are discussed in [24], section 2, and $\bar{\partial}$ -log compatible almost complex structures are discussed in section 3, where it is shown that the set of such almost complex structures tamed by ω is nonempty and contractible.

1.2. Outline of paper. Our gluing formulae follow from a natural fiber-product diagram

$$(3) \quad \begin{array}{ccc} \mathcal{M}_{[\gamma]^+}^{st} & \longrightarrow & \prod_e \bar{\mathbf{B}}_e \\ \downarrow \text{cut} & & \downarrow \Delta \\ \prod_v \mathcal{M}_{[\gamma_v^+]}^{st} & \longrightarrow & \prod_e \bar{\mathbf{B}}_e^2 \end{array}$$

discussed in Theorem 2.5. At the top left of the above diagram is a moduli stack of curves decorated by a tropical curve γ , and with an extra choice of point for each edge of γ . The downward arrow on the left cuts each such curve at the extra points to obtain cut-curves, and the right-pointing arrows indicate evaluation at those extra points on internal edges, and their cut analogues beneath. This fiber-product diagram, cut-curves and γ -decorated curves are the subject of section 2.

Diagram (3) does not immediately imply our gluing formulae. Because of the extra choice of points, the stacks $\mathcal{M}_{[\gamma]^+}^{st}$ and $\mathcal{M}_{[\gamma_v^+]}^{st}$ are bundles over the actual stacks $\mathcal{M}_{[\gamma]}^{st}$ and $\mathcal{M}_{[\gamma_v]}^{st}$ we need for our gluing formulae. To ‘forget the extra choice of points’ we consider a groupoid with objects parametrized by $\mathcal{M}_{[\gamma]^+}^{st}$, and quotient stack $\mathcal{M}_{[\gamma]}^{st}$. We show, in Proposition 3.6, that a diagram of groupoids analogous to (3) is also fiber-product diagram, however passing to quotient stacks does not quite give a fiber-product diagram. This technical hurdle is overcome using the tool of tropical completion in Lemma 4.4, after which our gluing formulae follow quickly from the machinery in [23, 25].

Section 3 constructs the evaluation map ev relevant for our first gluing formula. To relate to our natural fiber product diagram (3), we also construct an evaluation map \tilde{ev} on the level of groupoids, inducing a natural evaluation map \mathbf{ev} on the level of quotient stacks. Section 4 contains the proof of our first gluing formula, which is the result of Theorem 4.8 and Lemma 4.9. In section 5, we enhance our evaluation maps \mathbf{ev} and \tilde{ev} to obtain evaluation maps EV and \tilde{EV} that include information about the complex structure of the domain of curves. We then prove our second gluing formula, in Theorem 5.2 and Lemma 5.3. Section 5 concludes with an example showing how Kontsevich and Mannin’s splitting and genus-reduction axioms follow from our second gluing formula. The final section of this paper sketches how to extend our gluing formulae to keep track of more topological information, as, for example, is done in [6] using rim tori.

2. GLUING CUT-CURVES

In this section, we define various moduli stacks of (not-necessarily holomorphic) curves, including curves decorated by a tropical curve γ , and cut-curves. The section concludes with Theorem 2.5, which concerns a natural fiber-product diagram produced by gluing cut-curves back together. This theorem will be key for proving our tropical gluing formulae.

2.1. The moduli stack of γ -decorated curves.

Use the notation $\mathcal{M}_{\bullet}^{st}\mathbf{B}$ for a moduli stack of stable, (not necessarily holomorphic) decorated¹⁵ curves in \mathbf{B} , where the \bullet indicates some unspecified decoration. We shall also have use for possibly unstable curves. Use $\mathcal{M}_{\bullet}^{\infty,1}\mathbf{B}$ for the moduli stack of (not necessarily stable or holomorphic) decorated curves in \mathbf{B} .¹⁶ We shall use \mathcal{M}_{\bullet} in statements that hold for both $\mathcal{M}_{\bullet}^{st}$ and $\mathcal{M}_{\bullet}^{\infty,1}$.

¹⁵We use decorated moduli stack in the sense of Definition 2.12 of [23].

¹⁶The notation $\mathcal{M}^{\infty,1}$ indicates that we are working with exploded manifolds and maps of regularity $C^{\infty,1}$, which is as good as smooth for all practical purposes.

Let γ be a tropical curve in $\underline{\mathbf{B}}$. Let $\mathcal{M}_\gamma^\bullet$ indicate the moduli stack of curves within \mathcal{M}^\bullet with a chosen isomorphism of their tropical part to γ . Below we shall define a decorated moduli stack $\mathcal{M}_{[\gamma]}^\bullet$ that is a kind of closure of $\mathcal{M}_\gamma^\bullet$. Loosely, $\mathcal{M}_{[\gamma]}^\bullet$ is the closure of $\mathcal{M}_\gamma^\bullet$ in the moduli stack of curves f in \mathbf{B} with tropical part suitably labeled by γ .

Definition 2.1 (γ -decoration). *A γ -decorated tropical curve is a tropical curve in $\underline{\mathbf{B}}$ with a continuous affine map of its domain to the domain of γ so that this map is a homeomorphism restricted to the inverse image of the interior of all edges of γ and is an integral-affine isomorphism restricted to all exterior edges.*

Define a γ -decorated curve to be a curve with a γ -decorated tropical part. Consider the stack of γ -decorated curves, with objects $C^{\infty,1}$ families of curves $\hat{f} \in \mathcal{M}^\bullet(\mathbf{B})$ with an affine map of $\mathbf{C}(\hat{f})$ to the domain of γ that makes each individual curve a γ -decorated curve. $\mathcal{M}_\gamma^\bullet$ is a substack of this stack of γ -decorated curves. Define $\mathcal{M}_{[\gamma]}^\bullet$ to be the closure of $\mathcal{M}_\gamma^\bullet$ in this stack of γ -decorated curves.

To prove our gluing formula, we consider the following bundle over $\mathcal{M}_{[\gamma]}^\bullet$.

Definition 2.2. *Define $\mathcal{M}_{[\gamma]^+}^\bullet$ to be the stack of curves in $\mathcal{M}_{[\gamma]}^\bullet$ with an additional choice of point in each edge labeled by an edge of γ . So, a family in $\mathcal{M}_{[\gamma]^+}^\bullet$ is a family \hat{f} in $\mathcal{M}_{[\gamma]}^\bullet$, along with, for each edge e of γ , a $C^{\infty,1}$ section*

$$s_e: \mathbf{F}(\hat{f}) \longrightarrow \mathbf{C}(\hat{f})$$

with image contained in the edge labeled by e .

Composing s_e with \hat{f} defines a $C^{\infty,1}$ evaluation map

$$ev_e: \mathcal{M}_{[\gamma]^+}^\bullet \longrightarrow \mathbf{B}$$

Let $\bar{\mathbf{B}}_e$ indicate the closure of the strata of \mathbf{B} with tropical part containing the edge e . The evaluation map ev_e always lands in $\bar{\mathbf{B}}_e$.

$$ev_e: \mathcal{M}_{[\gamma]^+}^\bullet \longrightarrow \bar{\mathbf{B}}_e$$

This stack $\mathcal{M}_{[\gamma]^+}^\bullet$ is a bundle over $\mathcal{M}_{[\gamma]}^\bullet$. In particular, given any family \hat{f} in $\mathcal{M}_{[\gamma]}^\bullet$, the pullback of \hat{f} to $\mathcal{M}_{[\gamma]^+}^\bullet$ is a family \hat{f}^+ constructed as follows: Let $\mathbf{C}_e(\hat{f})$ indicate the strata of $\mathbf{C}(\hat{f})$ decorated by e . The domain $\mathbf{F}(\hat{f}^+)$ is the fiber-product of all these $\mathbf{C}_e(\hat{f})$ over $\mathbf{F}(\hat{f})$, the family \hat{f}^+ is the pullback of \hat{f} under the map $\mathbf{F}(\hat{f}^+) \longrightarrow \mathbf{F}(\hat{f})$, and s_e is given by the canonical map $\mathbf{F}(\hat{f}^+) \longrightarrow \mathbf{C}_e(\hat{f})$.

The family \hat{f}^+ is the pullback of \hat{f} to $\mathcal{M}_{[\gamma]^+}^\bullet$ in the sense that it satisfies the following universal condition. Let $\pi: \mathcal{M}_{[\gamma]^+}^\bullet \longrightarrow \mathcal{M}_{[\gamma]}^\bullet$ indicate the map that forgets the extra sections s_e . Given any family \hat{g} in $\mathcal{M}_{[\gamma]^+}^\bullet$ and a map $\pi(\hat{g}) \longrightarrow \hat{f}$, there exists a unique lift of this map making the diagram below commute.¹⁷

$$(4) \quad \begin{array}{ccc} \hat{g} & \xrightarrow{\exists!} & \hat{f}^+ \\ \downarrow \pi & & \downarrow \pi \\ \pi(\hat{g}) & \xrightarrow{\quad} & \pi(\hat{f}^+) \\ & \searrow & \downarrow \\ & & \hat{f} \end{array}$$

Both $\mathbf{C}_e(\hat{f})$ and $\bar{\mathbf{B}}_e$ have a (not everywhere defined) action of \mathbf{T} so that ev_e is equivariant. In section 3, we encode this action as a groupoid structure on $\mathbf{C}_e(\hat{f})$

¹⁷ Unlike the other arrows in the above diagram, the arrows π are not maps of families of curves.

and $\bar{\mathbf{B}}_e$, and extend ev_e to a map of groupoids. Taking quotients will then define an evaluation map from $\mathcal{M}_{[\gamma]}^\bullet$ instead of $\mathcal{M}_{[\gamma]^+}^\bullet$.

2.2. Cut-curves. The goal of this section is to prove that $\mathcal{M}_{[\gamma]^+}^\bullet$ is a fiber-product of some moduli stacks of cut-curves. For this, we shall be assuming that the domain of γ is connected, and not \mathbb{R} — in this exceptional case, $\mathcal{M}_{[\gamma]}^\bullet$ is easily studied directly. We can cut any curve in $\mathcal{M}_{[\gamma]^+}^\bullet$ at the extra points on its edges to obtain cut-curves. Cut-curves were defined in [22] omitting the definition of a family of cut-curves, given below.

Definition 2.3. *A family of cut-curves over an exploded manifold \mathbf{F} is*

- (1) *an abstract exploded space¹⁸ \mathbf{C} with a map of abstract exploded spaces $\mathbf{C} \rightarrow \mathbf{F}$,*
- (2) *some number of sections $s_e: \mathbf{F} \rightarrow \mathbf{C}$ called cuts so that \mathbf{C} minus the image of these sections is an exploded manifold,*
- (3) *a fiberwise almost complex structure j on \mathbf{C} minus all cuts,*

so that the above data is the result of trimming some family of curves (\mathbf{C}', j) over \mathbf{F} at sections $s_e: \mathbf{F} \rightarrow \mathbf{C}'$ with image in distinct ends of curves in \mathbf{C}' ; here ‘trimming’ means that the above data is the restriction of $(\mathbf{C}', j, \{s_e\})$ to the subset of \mathbf{C}' consisting of the image of the sections s_e and all points in \mathbf{C}' that have tropical part not as far out on any edge as the tropical part of the image of s_e .

A family of (holomorphic) cut-curves in \mathbf{B} is a (fiberwise holomorphic) map $\mathbf{C} \rightarrow \mathbf{B}$.

Although the domain of a cut-curve is the result of trimming some honest curve, it is not true that every cut-curve in \mathbf{B} can be obtained by trimming a curve in \mathbf{B} ; this is the case if and only if all the cut edges of the corresponding tropical curve extend semi-infinitely in \mathbf{B} .

Each cut-curve f in \mathbf{B} has a tropical part \underline{f} with special 1-valent vertices at the cuts. Call such a tropical curve a cut tropical curve. We may define a moduli space $\mathcal{M}_{[\gamma]}^\bullet$ of γ -decorated cut-curves as in Definition 2.1.

Given a tropical curve γ with a choice of point on each edge, we can obtain a (possibly disconnected) cut tropical curve by cutting γ at these chosen points, and discarding¹⁹ cut edges not attached to a vertex v . This cut-curve will have one connected component γ_v^+ for each vertex v of γ , so we can write this cut-curve as $\coprod_v \gamma_v^+$.

Lemma 2.4. *There exists a map of stacks*

$$\text{cut}: \mathcal{M}_{[\gamma]^+}^\bullet \rightarrow \prod_v \mathcal{M}_{[\gamma_v^+]}^\bullet.$$

In particular, given a family of curves \hat{f} in $\mathcal{M}_{[\gamma]^+}^\bullet$, there exists a family of curves $\text{cut } \hat{f}$ in $\prod_v \mathcal{M}_{[\gamma_v^+]}^\bullet$ and a morphism $\mathbf{C}(\text{cut } \hat{f}) \rightarrow \mathbf{C}(\hat{f})$ so that the following diagram

¹⁸ See [21], Definition 3.1. An abstract exploded space is a topological space with a sheaf of $\mathbb{C}^* \mathfrak{t}^{\mathbb{R}}$ -valued functions. One way of defining an abstract exploded space is as a subset of an exploded manifold \mathbf{B} given the subspace topology and the pullback of the sheaf of functions $\mathcal{E}^\times(\mathbf{B})$.

¹⁹In our neglected exceptional case that the domain of γ is \mathbb{R} , this cutting process would discard everything.

commutes

$$\begin{array}{ccccc}
 & & \text{cut } \hat{f} & & \\
 & \searrow & & \nearrow & \\
 \mathbf{C}(\text{cut } \hat{f}) & \longrightarrow & \mathbf{C}(\hat{f}) & \xrightarrow{\hat{f}} & \mathbf{B} \\
 s_{e_i} \uparrow \downarrow & & s_e \uparrow \downarrow & & \\
 \mathbf{F}(\text{cut } \hat{f}) & \xrightarrow{\cong} & \mathbf{F}(\hat{f}) & &
 \end{array}$$

where each internal edge e of γ corresponds to two cut edges, e_1 and e_2 , of $\coprod_v \gamma_v$, (and an end of γ corresponds to a unique cut edge of $\coprod_v \gamma_v$). Moreover, any morphism $x : \mathbf{C}(\text{cut } \hat{f}) \rightarrow \mathbf{A}$ is the pullback of a morphism $x' : \mathbf{C}(\hat{f}) \rightarrow \mathbf{A}$ if and only if for all internal edges e of γ , $x \circ e_1 = x \circ e_2$ and the derivative of \underline{x} along the edges e_i is opposite, and for all ends of γ , \underline{x} is infinitely extendible along the corresponding edge of $\coprod_v \gamma_v$.

Proof:

The idea is to cut our family \hat{f} at the sections $s_e : \mathbf{F}(\hat{f}) \rightarrow \mathbf{C}(\hat{f})$ to obtain a family of cut-curves (still parametrized by $\mathbf{F}(\hat{f})$) with one connected component \hat{f}_v in $\mathcal{M}_{[\gamma_v^+]}^\bullet$ for each vertex v of γ .

If there are no loops attached to v , we can define $\mathbf{C}(\hat{f}_v)$ as a subset of $\mathbf{C}(\hat{f})$ with the induced exploded structure. Namely, $\mathbf{C}(\hat{f}_v)$ is the union of the image of the sections $s_e : \mathbf{F}(\hat{f}) \rightarrow \mathbf{C}(\hat{f})$ for all edges e adjacent to v , and all points in \mathbf{C} with tropical part fiberwise closer to v than these sections. It is easily checked that $\mathbf{C}(\hat{f}_v) \rightarrow \mathbf{F}(\hat{f})$ along with these sections s_e , and the restriction of \hat{f} to $\mathbf{C}(\hat{f}_v)$ is a family of cut-curves. The above procedure fails in the case that there is an edge e with two ends attached to v , as we need to break apart the two halves of this edge.

In the general case, define the set $\mathbf{C}(\hat{f}_v)$ as the union of a copy of \mathbf{F} for each oriented edge leaving v with the subset of $\mathbf{C}(\hat{f})$ with tropical part sent closer to v than s_e . $\mathbf{C}(\hat{f}_v)$ comes with a natural map to $\mathbf{C}(\hat{f})$ which is the identity inclusion on the main part, and s_e on each copy of \mathbf{F} corresponding to an edge e .

Before continuing with the description of $\mathbf{C}(\hat{f}_v)$, consider the case of an individual curve f in \hat{f} . The γ -decoration gives a map $\mathbf{C}(f) \rightarrow \gamma$, and the image of s_e under this map is sent to a point on the edge e . Cut γ at these points, and consider the connected component γ_v^+ containing v . As a set, $\mathbf{C}(f_v)$ is the inverse image of the interior of γ_v^+ with a point for each oriented edge leaving v ; the identity inclusion together with the maps s_e defines a natural map $\mathbf{C}(f_v) \rightarrow \mathbf{C}(f)$ that is injective everywhere apart from the extra points corresponding to an edge e with both ends attached to v . With this understood, we could equivalently define $\mathbf{C}(\hat{f}_v)$ as the union of $\mathbf{C}(f_v)$ for all f in v . Note that $\mathbf{C}(f_v)$ comes with a surjective map to γ_v^+ so that the following diagram commutes.

$$\begin{array}{ccc}
 \mathbf{C}(f_v) & \longrightarrow & \mathbf{C}(f) \\
 \downarrow & & \downarrow \\
 \gamma_v^+ & \longrightarrow & \gamma
 \end{array}$$

The point corresponding to an oriented edge leaving e is sent to the cut endpoint of that edge in γ_v^+ , and elsewhere, the map $\mathbf{C}(f_v) \rightarrow \gamma_v^+$ is uniquely determined by the above diagram. The topology on $\mathbf{C}(f_v)$ is the topology generated by the inverse image of open sets from $\mathbf{C}(f)$, and the inverse image of the closure of any edge of γ_v^+ .

We can describe the topology on $\mathbf{C}(\hat{f}_v)$ similarly. For other curves f in \hat{f} , the corresponding γ_v^+ may have different edge lengths, however there is a canonical

piecewise-linear isomorphism between all these γ_v^+ , so we obtain a γ_v^+ -decoration map $\mathbf{C}(\hat{f}_v) \rightarrow \gamma_v^+$. The topology on $\mathbf{C}(\hat{f}_v)$ is the topology generated by the inverse image of open subsets of $\mathbf{C}(\hat{f})$ and the inverse image of the closure of any edge in γ_v^+ . With this topology, we can define the sheaf of exploded functions as follows: for any open subset U small enough that its closure only intersects (the inverse image of) one edge of γ_v^+ , the exploded functions are the $\mathbb{C}^*\mathfrak{t}^{\mathbb{R}}$ -valued functions pulled back from exploded functions from $\mathbf{C}(\hat{f})$ defined some open neighborhood of the image of U .

Note that as exploded functions on $\mathbf{C}(\hat{f}_v)$ are locally the pullback of exploded functions on \mathbf{C} , any morphism $x : \mathbf{C}(\text{cut } \hat{f}) \rightarrow \mathbf{A}$ is the pullback of a morphism $x' : \mathbf{C}(\hat{f}) \rightarrow \mathbf{A}$ if and only if for all internal edges e of γ , $x \circ e_1 = x \circ e_2$ and the derivative of \underline{x} along the edges e_i is opposite, and for all ends of γ , \underline{x} is infinitely extendible along the corresponding edge of $\coprod_v \gamma_v$.

With this definition, it is clear that $\mathbf{C}(\hat{f}_v) \rightarrow \mathbf{C}(\hat{f})$ is a morphism of abstract exploded manifolds, and that this construction is functorial: given any morphism $\hat{g} \rightarrow \hat{f}$ in $\mathcal{M}_{[\gamma]}^\bullet$, there is a unique morphism $\mathbf{C}(\hat{g}_v) \rightarrow \mathbf{C}(\hat{f}_v)$ so that the following diagram commutes.

$$\begin{array}{ccc} \mathbf{C}(\hat{g}_v) & \longrightarrow & \mathbf{C}(\hat{f}_v) \\ \downarrow & & \downarrow \\ \mathbf{C}(\hat{g}) & \longrightarrow & \mathbf{C}(\hat{f}) \end{array}$$

For each oriented edge e of γ_v^+ , there is a unique map (of sets)

$$s_e : \mathbf{F}(\hat{f}_v) := \mathbf{F}(\hat{f}) \rightarrow \mathbf{C}(\hat{f}_v)$$

that is a section of $\mathbf{C}(\hat{f}_v) \rightarrow \mathbf{F}(\hat{f}_v)$ with image in the inverse image of the cut end of e in γ_v^+ . The fact that the diagram below commutes implies that s_e is a morphism of abstract exploded manifolds; these morphisms s_e define the cuts of our family of cut-curves.

$$\begin{array}{ccc} \mathbf{C}(\hat{f}_v) & \longrightarrow & \mathbf{C}(\hat{f}) \\ s_e \uparrow & & s_e \uparrow \\ \mathbf{F}(\hat{f}_v) & \longrightarrow & \mathbf{F}(\hat{f}) \end{array}$$

The rest of the data for defining \hat{f}_v as a family of cut-curves is the fiberwise almost complex structure pulled back from $\mathbf{C}(\hat{f})$, and the map $\hat{f}_v : \mathbf{C}(\hat{f}_v) \rightarrow \mathbf{B}$ that is the pullback of $\hat{f} : \mathbf{C}(\hat{f}) \rightarrow \mathbf{B}$. With this definition it is easy to verify that given any morphism $\hat{g} \rightarrow \hat{f}$ in $\mathcal{M}_{[\gamma]}^\bullet$, the unique maps $\mathbf{C}(\hat{g}_v) \rightarrow \mathbf{C}(\hat{f}_v)$ above indeed define a unique morphism $\text{cut } \hat{g} \rightarrow \text{cut } \hat{f}$ compatible with all this structure, so our construction is functorial. With the functoriality of the construction understood, it is also easy to verify locally that the resulting \hat{f}_v is indeed a family of cut-curves in $\mathcal{M}_{[\gamma^+]}^\bullet$ satisfying Definition 2.3.

We have therefore defined a map of stacks,

$$\text{cut} : \mathcal{M}_{[\gamma]^+}^\bullet(\mathbf{B}) \rightarrow \prod_v \mathcal{M}_{[\gamma_v^+]}^\bullet(\mathbf{B})$$

where $\text{cut}(\hat{f}) := \coprod_v \hat{f}_v$, the (possibly disconnected) family of cut-curves parametrized by $\mathbf{F}(\hat{f})$ with domain $\coprod_v \mathbf{C}(\hat{f}_v)$.

□

For the following theorem, use the notation $\text{ied}\gamma$ for the set of internal edges of γ . For each $e \in \text{ied}\gamma$, there are two corresponding cut edges e_1, e_2 of $\coprod_v \gamma_v^+$ where we can define evaluation maps ev_{e_i} .

Theorem 2.5. *The following is a fiber-product diagram.*

$$\begin{array}{ccc} \mathcal{M}_{[\gamma]^+}^\bullet & \xrightarrow{\prod_{e \in \text{ied}\gamma} ev_e} & \prod_e \bar{\mathbf{B}}_e \\ \downarrow \text{cut} & & \downarrow \Delta \\ \prod_v \mathcal{M}_{[\gamma_v^+]}^\bullet & \xrightarrow{\prod_{e \in \text{ied}\gamma} (ev_{e_1}, ev_{e_2})} & \prod_e \bar{\mathbf{B}}_e^2 \end{array}$$

More precisely, given any family of cut-curves \hat{f} in $\prod_v \mathcal{M}_{[\gamma_v^+]}^\bullet$ so that for all internal edges e of γ , $ev_{e_1} = ev_{e_2}$ on \hat{f} , there exists a family $\text{cut}^* \hat{f}$ in $\mathcal{M}_{[\gamma]^+}^\bullet$ with an isomorphism $\text{cut}(\text{cut}^* \hat{f}) \rightarrow \hat{f}$. This family $\text{cut}^* \hat{f}$ satisfies the universal property that given any other family of curves \hat{g} in $\mathcal{M}_{[\gamma]^+}^\bullet$ with a map $\text{cut} \hat{g} \rightarrow \hat{f}$, there exists a unique map $\hat{g} \rightarrow \text{cut}^* \hat{f}$ so that the following diagram commutes:

$$\begin{array}{ccc} \hat{g} & \xrightarrow{\exists!} & \text{cut}^* \hat{f} \\ \downarrow \text{cut} & & \downarrow \text{cut} \\ \text{cut} \hat{g} & \xrightarrow{\quad} & \text{cut}(\text{cut}^* \hat{f}) \\ & \searrow & \downarrow \simeq \\ & & \hat{f} \end{array}$$

Proof:

The family $\text{cut}^* \hat{f}$ is constructed by gluing together \hat{f} at the matching cuts s_{e_1} and s_{e_2} . One way to describe $\mathbf{C}(\text{cut}^* \hat{f})$ is as follows:

- The set of points in $\mathbf{C}(\text{cut}^* \hat{f})$ is the union of $\mathbf{C}(\hat{f}) \setminus (\bigcup_{e_i} s_{e_i}(\mathbf{F}(\hat{f})))$ with $\mathbb{C}^* \times \mathbf{F}(\hat{f})$ for each internal edge e , and $\mathbb{C}^* \mathfrak{t}^{[0, \infty)} \times \mathbf{F}(\hat{f})$ for each end of γ . Write these extra points (c, p) as $c * s_e(p)$.
- There is a canonical map $\mathbf{C}(\hat{f}) \rightarrow \mathbf{C}(\text{cut}^* \hat{f})$: this map is the identity on $\mathbf{C}(\text{cut}^* \hat{f})$ minus all cuts, and $s_{e_i}(p) \mapsto 1 * s_e(p)$ on cuts.
- $\mathbf{C}(\text{cut}^* \hat{f})$ has the following topology: The open subsets $U \subset \mathbf{C}(\text{cut}^* \hat{f})$ have open inverse image in $\mathbf{C}(\hat{f})$ and satisfy the additional conditions that $c * s_e(p) \in U$ if and only if $1 * s_e(p) \in U$, and $1 * s_{e_1}(p) \in U$ if and only if $1 * s_{e_2}(p) \in U$.
- The defining sheaf of exploded functions on $\mathbf{C}(\text{cut}^* \hat{f})$ is as follows. The exploded functions x on $U \subset \mathbf{C}(\text{cut}^* \hat{f})$ are those that pull back to exploded functions on $\mathbf{C}(\hat{f})$, and satisfy the following addition conditions
 - $x \circ s_{e_1} = x \circ s_{e_2}$
 - The derivative of \underline{x} on the edges e_1 and e_2 is opposite — in other words, if \tilde{z}_i indicates standard coordinates on these edges, then $\tilde{z}_1 \frac{\partial}{\partial z_1} x = \alpha x$ and $\tilde{z}_2 \frac{\partial}{\partial z_2} x = -\alpha x$.
 - If α indicates the derivative of \underline{x} on the edge e , then

$$x(c * s_e(p)) := c^\alpha x(s_{e_i}(p)) .$$

The above implies that maps x from $\mathbf{C}(\text{cut}^* \hat{f})$ to any exploded manifold canonically correspond to maps x from $\mathbf{C}(\hat{f})$ so that $x \circ s_{e_1} = x \circ s_{e_2}$, the derivative of \underline{x} on the edges e_1 and e_2 is opposite, and the remaining

external edges of tropical curves in \underline{x} can be extended to be semi-infinite. In particular, the map $\mathbf{C}(\hat{f}) \rightarrow \mathbf{F}(\hat{f})$ induces a map $\mathbf{C}(\text{cut}^* \hat{f}) \rightarrow \mathbf{F}(\hat{f}) = \mathbf{F}(\text{cut}^* \hat{f})$. It also follows that the section $s_e: \mathbf{F}(\text{cut}^* \hat{f}) \rightarrow \mathbf{C}(\text{cut}^* \hat{f})$ pulls back exploded functions to exploded functions, so is a valid map of exploded manifolds.

To see that the above defines an exploded manifold structure on $\mathbf{C}(\text{cut}^* \hat{f})$, it suffices to check locally around the section s_e . We do this for e an internal edge, the argument for an external edge is similar but easier. Around any point in $\mathbf{F}(\hat{f})$ there exists a coordinate chart U so that a neighborhood of $s_{e_i}(U)$ is isomorphic to the result of trimming $U \times \mathbf{T}_1^1$ by sections $s_{e_i}: U \rightarrow \mathbf{T}_1^1$. Then, there is an open neighborhood V of $s_e(U) \subset \mathbf{C}(\text{cut}^* \hat{f})$ with topology and exploded functions the same as the following fiber-product:

$$\begin{array}{ccc} V & \longrightarrow & U \times \mathbf{T}_1^1 \times \mathbf{T}_1^1 \\ \downarrow & & \downarrow (u, \tilde{z}_1, \tilde{z}_2) \rightarrow (u, \tilde{z}_1 \tilde{z}_2) \\ U & \xrightarrow{s_{e_1} s_{e_2}} & U \times \mathbf{T}_1^1 \end{array}$$

Therefore, $\mathbf{C}(\text{cut}^* \hat{f})$ as defined above really is an exploded manifold. The fiberwise almost complex structure defining $\mathbf{C}(\text{cut}^* \hat{f}) \rightarrow \mathbf{F}(\text{cut}^* \hat{f})$ as a family of curves is induced from $\mathbf{C}(\hat{f}) \rightarrow \mathbf{F}(\hat{f})$ so that the inclusion $\mathbf{C}(\hat{f}) \rightarrow \mathbf{C}(\text{cut}^* \hat{f})$ is fiberwise holomorphic.

We have already assumed that $\hat{f}: \mathbf{C}(\hat{f}) \rightarrow \mathbf{B}$ satisfies the condition that $\hat{f} \circ s_{e_1} = \hat{f} \circ s_{e_2}$, so Lemma 2.4 implies that to verify that \hat{f} induces a map $\mathbf{C}(\text{cut}^* \hat{f}) \rightarrow \mathbf{B}$, it suffices to check that the derivatives of \hat{f} along the edges e_1 and e_2 are opposite, and that the remaining external edges of the tropical curves in \hat{f} extend semi-infinitely. These conditions hold if \hat{f} is in $\prod_v \mathcal{M}_{\gamma_v^+}^\bullet$, and therefore hold for \hat{f} in $\prod_v \mathcal{M}_{[\gamma_v^+]}^\bullet$, because $\prod_v \mathcal{M}_{\gamma_v^+}^\bullet$ is dense in $\prod_v \mathcal{M}_{[\gamma_v^+]}^\bullet$. Our map \hat{f} therefore induces our map $\text{cut}^* \hat{f}$.

$$\begin{array}{ccccc} \mathbf{C}(\hat{f}) & \longrightarrow & \mathbf{C}(\text{cut}^* \hat{f}) & \xrightarrow{\text{cut}^* \hat{f}} & \mathbf{B} \\ & \searrow \hat{f} & & \nearrow & \end{array}$$

Note that $\text{cut}^* \hat{f}$ consists of stable curves if and only if \hat{f} consists of stable curves. There is a unique isomorphism $\text{cut}(\text{cut}^* \hat{f}) \rightarrow \hat{f}$ compatible with our two canonical maps $\mathbf{C}(\hat{f}) \rightarrow \mathbf{C}(\text{cut}^* \hat{f})$ and $\mathbf{C}(\text{cut}(\text{cut}^* \hat{f})) \rightarrow \mathbf{C}(\text{cut}^* \hat{f})$.

Now we check the claimed universal property of $\text{cut}^* \hat{f}$. Given a family \hat{g} with a map $\text{cut} \hat{g} \rightarrow \hat{f}$ in $\prod_v \mathcal{M}_{[\gamma_v^+]}^\bullet$, the composition $\mathbf{C}(\text{cut} \hat{g}) \rightarrow \mathbf{C}(\hat{f}) \rightarrow \mathbf{C}(\text{cut}^* \hat{f})$ satisfies the conditions from Lemma 2.4 to be the pullback of a map $\mathbf{C}(\hat{g}) \rightarrow \mathbf{C}(\text{cut}^* \hat{f})$. This map defines a unique map $\hat{g} \rightarrow \text{cut}^* \hat{f}$ so that the required diagram commutes.

$$\begin{array}{ccc} \hat{g} & \xrightarrow{\exists!} & \text{cut}^* \hat{f} \\ \downarrow \text{cut} & & \downarrow \text{cut} \\ \text{cut} \hat{g} & \xrightarrow{\quad} & \text{cut}(\text{cut}^* \hat{f}) \\ & \searrow & \downarrow \simeq \\ & & \hat{f} \end{array}$$

□

The fiber-product diagram from Theorem 2.5 is equivariant with respect to some (partially defined) actions of \mathbf{T} corresponding to the extra choices of points on

curves in $\mathcal{M}_{[\gamma]^+}^\bullet$ and the length of cut edges in $\mathcal{M}_{[\gamma^+]}^\bullet$. We shall formalize this in the language of groupoids in the next section. For our gluing formula, and to define evaluation maps from $\mathcal{M}_{[\gamma]}^\bullet$ instead of $\mathcal{M}_{[\gamma]^+}^\bullet$, we must take the quotient by these \mathbf{T} actions.

3. THE SIMPLE EVALUATION MAP

Each end of a curve $\mathbf{C} \neq \mathbf{T}$ is isomorphic to $\mathbf{T}_{(0,\infty)}^1$, however there is a \mathbb{C}^* -fold choice of isomorphism. The moduli stack of maps $\mathbf{T}_{(0,\infty)}^1 \rightarrow \mathbf{B}$, identified up to isomorphism, is not a nice stack, so we shall replace it with a better behaved stack for defining our simple evaluation map. In particular, we throw away ‘tropical’ information by identifying two maps $\mathbf{T}_{(0,\infty)}^1 \rightarrow \mathbf{B}$ if they eventually coincide. By doing so, we obtain a better behaved stack $\mathcal{E}\mathbf{B}$, which is a quotient of an exploded manifold $\text{End } \mathbf{B}$ by some trivial group actions, and is an orbifold on most components.

In what follows, we shall describe $\mathcal{E}\mathbf{B}$ as the quotient stack of a Lie groupoid $\tilde{\mathcal{E}}\mathbf{B}$ consisting of a collection of connected components of a Lie groupoid $\mathcal{G}(\mathbf{B})$ with objects parametrized by the integral-vectors ${}^{\mathbb{Z}}T\mathbf{B}$.

We shall need various partially defined \mathbf{T} -actions, all in the form a ‘flow’ induced by an integral vector. Recall, from Definition 6.8 of [21], that integral tangent vectors ${}^{\mathbb{Z}}T\mathbf{B} \subset T\mathbf{B}$ are vectors v so that $v\tilde{z}$ is an integer times \tilde{z} for every (locally defined) exploded function \tilde{z} . For example, in standard coordinates on $\mathbf{T}_{[0,\infty)}^1$, the integral-vectors consist of the zero-vector, and integer multiples of the real part of $\tilde{z} \frac{\partial}{\partial \tilde{z}}$ on the strata over $(0, \infty)$, where $\tilde{z} = 0$. As a subset of the exploded manifold $T\mathbf{B}$, the set of integral tangent vectors inherits the structure of an exploded manifold. For example ${}^{\mathbb{Z}}T\mathbf{T}_{[0,\infty)}^1$ is isomorphic to the disjoint union of $\mathbf{T}_{[0,\infty)}^1$ with a copy of $\mathbf{T}_{(0,\infty)}^1$ for every nonzero integer. There is a natural (not everywhere defined) action of \mathbf{T} on ${}^{\mathbb{Z}}T\mathbf{B}$ so that for any exploded function \tilde{z} ,

$$(c * v)\tilde{z} = c^{\tilde{z}^{-1}v\tilde{z}}v\tilde{z}.$$

There may be no vector $c * v$ satisfying the above conditions, in which case $c * v$ is not defined. For example, if v is the real part of $\tilde{z} \frac{\partial}{\partial \tilde{z}}$ over the point where $\tilde{z} = 1\mathbf{t}^1$, $c * v$ is the same vector at the point $\tilde{z} = c \times \mathbf{t}^1$ so long as $c > -1$. For $c \leq -1$, $c * v$ is not defined. For $c \in \mathbb{C}^* \subset \mathbf{T}$, this action is always defined, and $e^t * v$ can be thought of as the result of flowing v for time t .

The above action defines a groupoid $\mathcal{G}(\mathbf{B})$ with objects $\mathcal{G}_0(\mathbf{B})$ parametrized by ${}^{\mathbb{Z}}T\mathbf{B}$, and morphisms $\mathcal{G}_1(\mathbf{B})$ parametrized by the set of (c, v) so that $c * v$ is defined. We shall see that $\mathcal{G}_1(\mathbf{B})$ inherits an exploded manifold structure as a subset of $\mathbf{T} \times \mathcal{G}_0(\mathbf{B})$. This groupoid is a nice Lie groupoid in the category of exploded manifolds, with all structure maps morphisms in the category of exploded manifolds. Moreover, \mathcal{G} defines a functor from the category of exploded manifolds to the category of such Lie groupoids.

To understand this functor \mathcal{G} , consider $\mathcal{G}(\mathbf{T}_P^m)$. For each integral-vectorfield v on P , there is a corresponding integral-vectorfield on the subset of \mathbf{T}_P^m with tropical part P_v the union of all strata of P tangent to v . Then

$$\mathcal{G}_0(\mathbf{T}_P^m) = \coprod_{v \in \mathbb{Z}^m} \mathbf{T}_{P_v}^m.$$

Morphisms in $\mathcal{G}(\mathbf{T}_P^m)$ always stay within these connected components, and $\mathcal{G}_1(\mathbf{T}_P^m)$ is also a union of connected components indexed by integral-vectors v . Define $\hat{P}_v \subset P_v \times \mathbb{R}$ to be

$$\hat{P}_v := \{(p, x) \in P_v \times \mathbb{R} \text{ so that } p + xv \in P_v\}.$$

The tropical part of $\mathcal{G}_1(\mathbf{T}_P^m)$ is the disjoint union of these \hat{P}_v .

$$\mathcal{G}_1(\mathbf{T}_P^m) = \coprod_{v \in \mathbb{Z}^m} \mathbf{T}_{\hat{P}_v}^{m+1}$$

There is a canonical inclusion of P_v into \hat{P}_v as $P_v \times 0$, corresponding to the ‘identity’ section $\text{id}: \mathbf{T}_{P_v}^m \rightarrow \mathbf{T}_{\hat{P}_v}^{m+1}$ sending $v \in \mathbf{T}_{P_v}^m$ to $(v, 1\mathbf{t}^1)$. \hat{P}_v comes with two surjective integral-affine maps to P_v : the projection $(p, x) \mapsto p$, and the map $(p, x) \mapsto p + xv$. These two maps are the tropical part of the two structure maps $s, t: \mathcal{G}_1 \rightarrow \mathcal{G}_0$.

$$\begin{aligned} s, t: \mathbf{T}_{\hat{P}_v}^{m+1} &\rightarrow \mathbf{T}_{P_v}^m \\ s(v, c) &:= v \\ t(v, c) &:= c * v \end{aligned}$$

Put the analogous exploded manifold structure on $\mathcal{G}(\mathbb{R}^n \times \mathbf{T}_P^m) = \mathbb{R}^n \times \mathcal{G}(\mathbf{T}_P^m)$. To put an exploded manifold structure on $\mathcal{G}_1(\mathbf{B})$, note that there is a natural projection $\mathcal{G}_0(\mathbf{B}) \rightarrow \mathbf{B}$, and that our action is always trivial on $[\mathbb{Z}T\mathbf{B}]$, so s and t followed by the smooth part of this projection define the same map $\mathcal{G}_1(\mathbf{B}) \rightarrow [\mathbf{B}]$. Put the pulled-back topology on $\mathcal{G}_1(\mathbf{B})$, and then give $\mathcal{G}(\mathbf{B})$ over a coordinate chart the exploded structure discussed above. This exploded structure on $\mathcal{G}(\mathbf{B})$ is well-defined, and \mathcal{G} defines a functor to the category of (exploded) Lie groupoids.

Remark 3.1. *The functor \mathcal{G} sends fiber-products to fiber-products.*

$$\mathcal{G}(\mathbf{A} \times_{\mathbf{C}} \mathbf{B}) = \mathcal{G}(\mathbf{A}) \times_{\mathcal{G}(\mathbf{C})} \mathcal{G}(\mathbf{B})$$

The special case of $\mathbf{A} \times \mathbf{B}$ must be considered as the fiber-product over a point; $\mathcal{G}(pt)$ is the groupoid with a single object, and morphisms parametrized by \mathbf{T} .

If we consider $\mathcal{G}(\mathbf{B})$ as a stack $\mathcal{BG}(\mathbf{B})$ (in other words, replace $\mathcal{G}(\mathbf{B})$ by the quotient stack classifying principle $\mathcal{G}(\mathbf{B})$ -bundles, as in Definition 3.17 of [9] translated to the category of exploded manifolds), then the components of \mathcal{BG} corresponding to nonzero integral-vectors are orbifolds,²⁰ and the components corresponding to primitive integral-vectors are exploded manifolds. The remaining zero-vector component is the quotient of \mathbf{B} by the trivial \mathbf{T} -action.

More explicitly, \mathcal{BG} consists of the stack of bundles $\mathbf{L} \rightarrow \mathbf{F}$ along with maps of groupoids,

$$\begin{array}{ccc} \mathbf{L} \times_{\mathbf{F}} \mathbf{L} & \xrightarrow{\phi_1} & \mathcal{G}_1 \\ \downarrow \pi_1 \text{ or } \pi_2 & & \downarrow s \text{ or } t \\ \mathbf{L} & \xrightarrow{\phi_0} & \mathcal{G}_0 \end{array}$$

so that the above diagrams are fiber-product diagrams. The above data defines an object in \mathcal{BG} , and a morphism is a commutative diagram

$$\begin{array}{ccc} \mathbf{L}_1 & \longrightarrow & \mathbf{L}_2 \\ \downarrow & & \downarrow \\ \mathbf{F}_1 & \longrightarrow & \mathbf{F}_2 \end{array}$$

compatible with the maps into \mathcal{G} .

For example, consider the component of $\mathcal{BG}(\mathbf{T}_P^m)$ corresponding to a nonzero integral-vector v . The v -component of $\mathcal{G}_0(\mathbf{T}_P^m)$ is $\mathbf{T}_{P_v}^m$. Each fiber of $\mathbf{L} \rightarrow \mathbf{F}$ must be isomorphic to some $\mathbf{T}_{(a,b)}^1$, and the map ϕ_0 to $\mathbf{T}_{P_v}^m$ restricted to each fiber must have tropical part with derivative v . Moreover, ϕ_0 restricted to each fiber must be

²⁰By an exploded orbifold, we mean a Deligne-Mumford stack in the category of exploded manifolds

complete. (The map ϕ_1 is uniquely determined by ϕ_0 .) Let P/v be the image of P_v under the projection with kernel spanned by v . There is a unique map $\mathbf{F} \rightarrow \mathbf{T}_{P/v}^{m-1}$ so that the following diagram commutes.

$$\begin{array}{ccc} \mathbf{L} & \xrightarrow{\phi_0} & \mathbf{T}_{P_v}^m \\ \downarrow & & \downarrow \\ \mathbf{F} & \longrightarrow & \mathbf{T}_{P/v}^{m-1} \end{array}$$

In the case that v is primitive, the above is a pullback diagram, and a family in this component of $\mathcal{BG}(\mathbf{T}_P^m)$ parametrized by \mathbf{F} is equivalent to a choice of such a map $\mathbf{F} \rightarrow \mathbf{T}_{P/v}^{m-1}$. The universal family over such a component is the projection $\mathbf{T}_{P_v}^m \rightarrow \mathbf{T}_{P/v}^{m-1}$. If v is k times a primitive integral-vector, the v -component of $\mathcal{BG}(\mathbf{T}_P^m)$ is the quotient of $\mathbf{T}_{P/v}^{m-1}$ by the trivial \mathbb{Z}_k action. In this case, the universal family over $\mathbf{T}_{P/v}^{m-1}$ has tropical part a polytope P'_v with a given integral-affine map $P'_v \rightarrow P_v$ that stretches a primitive vector to v , and is an affine isomorphism with determinant k . There is a corresponding degree- k map $\mathbf{T}_{P'_v}^m \rightarrow \mathbf{T}_{P_v}^m$ that gives a family

$$\begin{array}{ccc} \mathbf{T}_{P'_v}^m & \longrightarrow & \mathbf{T}_{P_v}^m \\ \downarrow & & \\ \mathbf{T}_{P/v}^{m-1} & & \end{array}$$

with automorphism group \mathbb{Z}_k (multiplying fibers by k th roots of unity.) Given any other family \mathbf{L} in this component of $\mathcal{BG}(\mathbf{T}_P^m)$, there is a unique \mathbb{Z}_k -fold cover, $\mathbf{L} \times_{\mathbf{T}_{P_v}^m} \mathbf{T}_{P'_v}^m$ of \mathbf{L} with a \mathbb{Z}_k -equivariant morphism to this family. Conversely, given a \mathbb{Z}_k -fold cover $\hat{\mathbf{F}}$ of \mathbf{F} and a \mathbb{Z}_k -invariant map $\psi: \hat{\mathbf{F}} \rightarrow \mathbf{T}_{P/v}^{m-1}$, we can construct a family parametrized by \mathbf{F} in this component of $\mathcal{BG}(\mathbf{T}_P^m)$ by taking the pullback of the above family by ψ , then taking the quotient by the \mathbb{Z}_k -action. In summary, this v -component of $\mathcal{BG}(\mathbf{T}_P^m)$ is the quotient of $\mathbf{T}_{P/v}^{m-1}$ by the trivial \mathbb{Z}_k -action, and the pullback of the universal family to $\mathbf{T}_{P/v}^{m-1}$ is the family displayed above.

Below, we shall often restrict interest to components of $\mathcal{G}(\mathbf{B})$ corresponding to vectors that span infinite rays in \mathbf{B} . We shall use the notation $\tilde{\mathcal{E}}(\mathbf{B})$ for these components of $\mathcal{G}(\mathbf{B})$, and $\mathcal{E}(\mathbf{B})$ for the quotient stack of $\tilde{\mathcal{E}}(\mathbf{B})$. The following lemma establishes that $\mathcal{E}(\mathbf{B})$ is complete if \mathbf{B} is complete; the same fails to hold for $\mathcal{BG}(\mathbf{B})$ in general.

Lemma 3.2. *If P is complete, and v spans an infinite ray in P , then P/v is complete.*

Proof: As P is complete, it is a subset of \mathbb{R}^m defined by some finite set of inequalities $\alpha_i \geq 0$ where $\alpha_i: \mathbb{R}^m \rightarrow \mathbb{R}$ is integral-affine. P_v consists of the union of all strata tangent to v , so P_v is defined by the equations $\alpha_i \geq 0$ and $\alpha_j > 0$ for all such α_j so that $v\alpha_j \neq 0$. As v spans an infinite ray in P , $v\alpha_i \geq 0$ for all i , therefore traveling in the direction of v , we can make all α_j so that $v\alpha_j \neq 0$ as large as we like. It follows that $P/v \subset \mathbb{R}^n/v$ is defined by the projection of the equations $\alpha_i \geq 0$ for α_i so that $v\alpha_i = 0$. Therefore, P/v is closed, as required. \square

Lemma 3.3. *\mathcal{BG} is a functor from the category of exploded manifolds to stacks over the category of exploded manifolds.*

Proof: As \mathcal{G} is a functor to the category of Lie groupoids, this lemma follows from the functoriality of passing from Lie groupoids to quotient stacks, given in [9]. Still, we should check this still works when exploded manifolds are used instead of smooth manifolds.

Given a map of exploded manifolds, $\psi: \mathbf{A} \rightarrow \mathbf{B}$, and a family

$$\begin{array}{ccc} \mathbf{L} \times_{\mathbf{F}} \mathbf{L} & \xrightarrow{\phi_1} & \mathcal{G}_1 \mathbf{A} \\ \downarrow & & \downarrow \\ \mathbf{L} & \xrightarrow{\phi_0} & \mathcal{G}_0 \mathbf{A} \\ \downarrow & & \\ \mathbf{F} & & \end{array}$$

in $\mathcal{BG}(\mathbf{A})$, composing ϕ_i with $\mathcal{G}_i \psi$ may not give a family in $\mathcal{BG}(\mathbf{B})$, because $\mathbf{L} \times_{\mathbf{F}} \mathbf{L}$ may not be the pullback of $\mathcal{G}_1 \mathbf{B}$, or equivalently in this case, $(\mathcal{G}_0 \psi) \circ \phi_0$ may not be complete restricted to each fiber of $\mathbf{L} \rightarrow \mathbf{F}$. There is, however, a unique extension of this bundle to a family in $\mathcal{BG}(\mathbf{B})$ constructed by lengthening the fibers of $\mathbf{L} \rightarrow \mathbf{F}$ until the map to $\mathcal{G}_0(\mathbf{B})$ is fiberwise complete. This is equivalently given by replacing \mathbf{L} by the quotient of $\mathbf{L} \times_{\mathcal{G}_0 \mathbf{B}} \mathcal{G}_1(\mathbf{B})$ by the diagonal action of $\mathcal{G}(\mathbf{A})$ on \mathbf{L} and $\mathcal{G}_1 \mathbf{B}$; see [9], Remarks 3.24 and 3.30 and the beginning of section 4. In our case, this means identifying $(l, c) \in \mathbf{L} \times_{\mathcal{G}_0(\mathbf{B})} \mathcal{G}_1 \mathbf{B}$ with $(c' * l, c'^{-1} c)$ for all c' so that $c' * l$ is defined. The map to $\mathcal{G}_0 \mathbf{B}$ sends (l, c) to $c * (\mathcal{G}_0 \psi \circ \phi_0)(l)$. The map from the square of this bundle to $\mathcal{G}_1(\mathbf{B})$ sends $(l, c_1, c_2 * l, c_3)$ to $(c_1 * (\mathcal{G}_0 \psi \circ \phi_0)(l), c_1^{-1} c_2 c_3)$. All this really does in our case is lengthen the fibers of \mathbf{L} until $\mathbf{L} \times_{\mathbf{F}} \mathbf{L}$ is the pullback of $\mathcal{G}_1(\hat{\mathbf{B}})$. The fact that such an extension is unique implies that this construction is compatible with morphisms between families in $\mathcal{BG}(\mathbf{A})$ so we obtain a functor $\mathcal{BG}(\mathbf{A}) \rightarrow \mathcal{BG}(\mathbf{B})$. Similarly, this uniqueness implies that this construction is compatible with composition of maps of exploded manifolds, so \mathcal{BG} defines a functor from the category of exploded manifolds to the category of stacks, as required. \square

Remark 3.4. *Some tropical information is discarded passing from \mathcal{G} to \mathcal{BG} so that it is not true in general that $\mathcal{BG}(\mathbf{A} \times_{\mathbf{C}} \mathbf{B})$ is equal to $\mathcal{BG}(\mathbf{A}) \times_{\mathcal{BG}(\mathbf{C})} \mathcal{BG}(\mathbf{B})$. An example is given by $\mathbf{A} = \mathbf{T}_{(0, \infty)}^1 \subset \mathbf{T} = \mathbf{C}$, $\mathbf{B} = \mathbf{T}_{(-\infty, 0)}^1 \subset \mathbf{T}$. This loss of tropical information is reflected in the need to lengthen the fibers of \mathbf{L} in the proof of Lemma 3.3 above.*

Definition 3.5. *Let $\mathcal{E}(\mathbf{B}) \subset \mathcal{BG}(\mathbf{B})$ be the substack consisting of connected components of $\mathcal{BG}(\mathbf{B})$ corresponding to vectors spanning infinite rays in $\underline{\mathbf{B}}$, and let $\tilde{\mathcal{E}}(\mathbf{B}) \subset \mathcal{G}(\mathbf{B})$ be the corresponding Lie groupoid.*

Given an integral-vector w in ${}^{\mathbb{Z}}T\mathbf{B}$ or the tangent space to $\underline{\mathbf{B}}$, let $\mathcal{G}_0^w(\mathbf{B}) \subset {}^{\mathbb{Z}}T\mathbf{B}$ be the closure of the stratum containing w , and let $\mathcal{G}^w(\mathbf{B})$ be the pullback of $\mathcal{G}(\mathbf{B})$ via the inclusion $\mathcal{G}_0^w(\mathbf{B}) \subset \mathcal{G}_0(\mathbf{B})$. Given a tropical curve γ in $\underline{\mathbf{B}}$, choose a primitive integral-vector on each edge e of the domain of γ , and let w_e be its image in $T\underline{\mathbf{B}}$. Then define the following Lie groupoid

$$\tilde{\mathcal{E}}^{[\gamma]}(\mathbf{B}) := \prod_e \mathcal{G}^{w_e}(\mathbf{B})$$

and let $\mathcal{E}^{[\gamma]}(\mathbf{B})$ be the quotient stack of $\tilde{\mathcal{E}}^{[\gamma]}(\mathbf{B})$.

Note that there is a canonical isomorphism $\mathcal{G}^w(\mathbf{B}) \rightarrow \mathcal{G}^{-w}(\mathbf{B})$ sending (v, c) to $(-v, c^{-1})$, so our construction of $\mathcal{E}^{[\gamma]}(\mathbf{B})$ does not depend on orientation of our primitive integral-vector on each edge. For compatibility with $\tilde{\mathcal{E}}(\mathbf{B})$, we always choose the outgoing primitive integral-vector on each end of γ .

Each connected component of $\mathcal{E}(\mathbf{B})$ or $\mathcal{E}^{[\gamma]}(\mathbf{B})$ is the quotient of an exploded manifold by a trivial group action. Forgetting this group action gives an exploded manifold $\text{End}(\mathbf{B})$ or $\text{End}^{[\gamma]}(\mathbf{B})$ respectively. For example,

$$\text{End}(\mathbf{T}_P^m) = \mathbf{T}_P^m \coprod_v \mathbf{T}_{P/v}^m$$

where the disjoint union is over all nonzero integral-vectors v that span an infinite ray in P . On the other hand,

$$\mathcal{E}(\mathbf{T}_P^m) = \mathbf{T}_P^m / \mathbf{T} \coprod_v \mathbf{T}_{P/v}^m / \mathbb{Z}_{|v|}$$

where $|v|$ is the positive integer so that $v/|v|$ is a primitive integral-vector, and the action of $\mathbb{Z}_{|v|}$ and \mathbf{T} is the trivial action.

For each end of γ , there is an outward-pointing integral-vector on the corresponding edge of γ , and a corresponding map $\tilde{\mathcal{E}}^{[\gamma]}(\mathbf{B}) \rightarrow \tilde{\mathcal{E}}(\mathbf{B})$. If γ has n ends, the product of these maps gives a map

$$\tilde{i}^{[\gamma]}: \tilde{\mathcal{E}}^{[\gamma]}(\mathbf{B}) \rightarrow (\tilde{\mathcal{E}}(\mathbf{B}))^n$$

which induces the following map on the level of quotient stacks.

$$i^{[\gamma]}: \mathcal{E}^{[\gamma]}(\mathbf{B}) \rightarrow (\mathcal{E}(\mathbf{B}))^n$$

So long as the domain of γ is not \mathbb{R} , this map $i^{[\gamma]}$ is a projection (onto the closure of some stratum) that forgets the components $\mathcal{G}^{w_e}(\mathbf{B})$ for internal edges e . The tropical completion of this map at the image of γ will feature in our gluing formula, but to reduce notational complexity, we shall refer to this tropical completion again as $i^{[\gamma]}$.

The definition of $\tilde{\mathcal{E}}^{[\gamma]}(\mathbf{B})$ also makes sense if γ is a cut tropical curve. Cut γ at some chosen point on all edges, and let γ_v^+ be the connected component containing v . As each edge in γ_v^+ is part of an edge in γ , there is a canonical map

$$\tilde{\Delta}: \tilde{\mathcal{E}}^{[\gamma]}(\mathbf{B}) \rightarrow \prod_v \tilde{\mathcal{E}}^{[\gamma_v^+]}(\mathbf{B}).$$

Now suppose that \mathbf{B} is basic, and let $\bar{\mathbf{B}}_v$ be the closure of the stratum of \mathbf{B} containing v . As \mathbf{B} is basic, there is an inclusion $\bar{\mathbf{B}}_v \subset \mathbf{B}|_v$, where $\mathbf{B}|_v$ is the tropical completion of \mathbf{B} at v discussed in [25]. The cut tropical curve γ_v^+ is contained in $\bar{\mathbf{B}}_v$, and $\tilde{\mathcal{E}}^{[\gamma_v^+]}(\mathbf{B})$ is a sub-groupoid of $\mathcal{G}(\bar{\mathbf{B}}_v)^n$, so the inclusion $\bar{\mathbf{B}}_v \subset \mathbf{B}|_v$ induces an inclusion $\tilde{\mathcal{E}}^{[\gamma_v^+]} \rightarrow \mathcal{G}(\mathbf{B}|_v)^n$. There is a tropical curve γ_v in $\mathbf{B}|_v$ with a single vertex v and all edges semi-infinite rays in the directions of the edges leaving v . The above inclusion has image contained in $\tilde{\mathcal{E}}^{[\gamma_v]}(\mathbf{B}|_v)$, so our map $\tilde{\Delta}$ above induces a canonical map

$$\tilde{\Delta}: \tilde{\mathcal{E}}^{[\gamma]}(\mathbf{B}) \rightarrow \prod_v \tilde{\mathcal{E}}^{[\gamma_v]}(\mathbf{B}|_v)$$

which induces the following map on the level of quotient stacks.

$$\Delta: \mathcal{E}^{[\gamma]}(\mathbf{B}) \rightarrow \prod_v \mathcal{E}^{[\gamma_v]}(\mathbf{B}|_v)$$

Again, it is the tropical completion of Δ at the image of γ that will feature in our gluing formula, but to reduce notational complexity, we shall refer to this tropical completion simply as Δ .

We now describe our evaluation maps to $\mathcal{E}(\mathbf{B})$. Given a connected family of curves \hat{f} in $\mathcal{M}^\bullet \mathbf{B}$ and an end-label, e , let $\tilde{\mathbf{F}}^e(\hat{f})$ be the connected component

of $\tilde{\mathcal{E}}\mathbf{C}(\hat{f})$ corresponding to the outgoing primitive integral-vector on the end e . Note that the quotient stack of $\tilde{\mathbf{F}}^e(\hat{f})$ is canonically isomorphic to $\mathbf{F}(\hat{f})$, so the restriction of $\mathcal{E}\hat{f}: \mathcal{E}(\mathbf{C}(\hat{f})) \rightarrow \mathcal{E}\mathbf{B}$ to this component defines a natural evaluation map $\mathbf{F}(\hat{f}) \rightarrow \mathcal{E}\mathbf{B}$. As \mathcal{E} is a functor, we get a corresponding evaluation map $\mathcal{M}^\bullet(\mathbf{B}) \rightarrow \mathcal{E}\mathbf{B}$ for each end e . The product of these evaluation maps for all labeled ends gives an evaluation map \mathbf{ev} , and composing this evaluation map with the map forgetting stack structure gives a further evaluation map, ev .

$$\begin{array}{c} \xrightarrow{\quad ev \quad} \\ \mathcal{M}^\bullet(\mathbf{B}) \xrightarrow{\quad \mathbf{ev} \quad} \coprod_n (\mathcal{E}\mathbf{B})^n \longrightarrow \coprod_n (\text{End } \mathbf{B})^n \end{array}$$

Let us lift this evaluation map \mathbf{ev} to give some kind of evaluation map to the corresponding groupoid $\coprod_n (\tilde{\mathcal{E}}\mathbf{B})^n$. Given a family of curves \hat{f} in $\mathcal{M}^\bullet\mathbf{B}$, let $\tilde{\mathbf{F}}(\hat{f})$ be the fiber-product of $\tilde{\mathbf{F}}^e(\hat{f})$ over $\mathbf{F}(\hat{f})$ for all ends e . Then $\tilde{\mathbf{F}}$ defines a functor from \mathcal{M}^\bullet to the category of (exploded) Lie groupoids. The quotient stack of the groupoid $\tilde{\mathbf{F}}(\hat{f})$ is equal to $\mathbf{F}(\hat{f})$, and $\mathbf{ev}: \mathbf{F}(\hat{f}) \rightarrow \coprod_n (\mathcal{E}\mathbf{B})^n$ lifts to a map of Lie groupoids induced from $\mathcal{G}\hat{f}$.

$$\tilde{\mathbf{ev}}: \tilde{\mathbf{F}}(\hat{f}) \rightarrow \coprod_n (\tilde{\mathcal{E}}\mathbf{B})^n$$

Clearly, $\tilde{\mathbf{ev}}$ is a natural transformation from the functor $\tilde{\mathbf{F}}$. We can also consider $\tilde{\mathbf{ev}}$ as a map of groupoids internal to the category of stacks over exploded manifolds. Let $\mathcal{M}_{,+}^\bullet$ be the moduli stack of curves in \mathcal{M}^\bullet along with a choice of point in every external edge. We can consider $\tilde{\mathbf{ev}}$ as given by the maps $\tilde{\mathbf{ev}}_i$ of stacks in the following commutative diagram

$$\begin{array}{ccccc} \mathbf{F}(\hat{f}^{++}) = \tilde{\mathbf{F}}_1(\hat{f}) & \longrightarrow & \mathcal{M}_{,++}^\bullet & \xrightarrow{\tilde{\mathbf{ev}}_1} & \coprod_n (\tilde{\mathcal{E}}_1\mathbf{B})^n \\ \Downarrow & & \Downarrow & & \Downarrow \\ \mathbf{F}(\hat{f}^+) = \tilde{\mathbf{F}}_0(\hat{f}) & \longrightarrow & \mathcal{M}_{,+}^\bullet & \xrightarrow{\tilde{\mathbf{ev}}_0} & \coprod_n (\tilde{\mathcal{E}}_0\mathbf{B})^n \\ \downarrow & & \downarrow & & \downarrow \\ \mathbf{F}(\hat{f}) & \longrightarrow & \mathcal{M}^\bullet & \xrightarrow{\mathbf{ev}} \coprod_n (\mathcal{E}\mathbf{B})^n \longrightarrow \coprod_n (\text{End } \mathbf{B})^n \\ & & & \searrow \quad \quad \quad \nearrow \\ & & & ev \end{array}$$

where $\mathcal{M}_{,++}^\bullet := \mathcal{M}_{,+}^\bullet \times_{\mathcal{M}^\bullet} \mathcal{M}_{,+}^\bullet$. The map $\tilde{\mathbf{ev}}_0$ is given by evaluation of $T\hat{f}$ at the outgoing primitive integral-vector at each of the points chosen on external edges. A curve in $\mathcal{M}_{,++}^\bullet$ has two chosen points in each external edge. In a family \hat{f} , this corresponds to two sections $\mathbf{F}(\hat{f}) \rightarrow \mathbf{C}(\hat{f})$ for each external edge e ; the second section is c_e times the first section for some exploded function $c_e: \mathbf{F}(\hat{f}) \rightarrow \mathbf{T}$. The map $\tilde{\mathbf{ev}}_1$ is the lift of $\tilde{\mathbf{ev}}_0$ determined by these functions c_e when we consider $\tilde{\mathcal{E}}_1(\mathbf{B}) \subset \tilde{\mathcal{E}}_0(\mathbf{B}) \times \mathbf{T}$. If we consider $\mathcal{M}_{,++}^\bullet \rightrightarrows \mathcal{M}_{,+}^\bullet$ a groupoid $\tilde{\mathcal{M}}^\bullet$ (internal to the category of stacks over exploded manifolds), the above maps $\tilde{\mathbf{ev}}_i$ can be represented as a map

$$\tilde{\mathbf{ev}}: \tilde{\mathcal{M}}^\bullet \rightarrow \coprod_n (\tilde{\mathcal{E}}\mathbf{B})^n$$

Given a tropical curve γ in \mathbf{B} , we can describe similar evaluation maps, $\mathbf{ev}^{[\gamma]}$, $ev^{[\gamma]}$ and $\tilde{\mathbf{ev}}^{[\gamma]}$, with target $\mathcal{E}^{[\gamma]}\mathbf{B}$, $\text{End}^{[\gamma]}\mathbf{B}$, and $\tilde{\mathcal{E}}^{[\gamma]}\mathbf{B}$. In this case, we need to use the stack $\mathcal{M}_{[\gamma]}^\bullet$ from Definition 2.1. Recall from Definition 2.2 and the discussion following it, that $\mathcal{M}_{[\gamma],+}^\bullet$ is the stack of curves in $\mathcal{M}_{[\gamma]}^\bullet$ along with a choice of point

in each edge decorated by an edge of γ , and that a family \hat{f} in $\mathcal{M}_{[\gamma]}^\bullet$ has a canonical lift to a family \hat{f}^+ in $\mathcal{M}_{[\gamma]}^\bullet$ parametrized by the fiber-product over $\mathbf{F}(\hat{f})$ of the strata $\mathbf{C}_e \subset \mathbf{C}(\hat{f})$ decorated by edges of γ . Define $\tilde{\mathbf{F}}_0(\hat{f}) := \mathbf{F}(\hat{f}^+)$ and let $\tilde{\mathbf{F}}(\hat{f})$ be the banal groupoid of the bundle $\tilde{\mathbf{F}}_0(\hat{f}) \rightarrow \mathbf{F}(\hat{f})$, so $\tilde{\mathbf{F}}_1 = \tilde{\mathbf{F}}_0 \times_{\mathbf{F}} \tilde{\mathbf{F}}_0$. If γ has n edges, choosing an orientation for each edge of γ identifies $\tilde{\mathbf{F}}(\hat{f})$ as a sub-groupoid of $(\mathcal{GC}(\hat{f}))^n$. Then $\mathcal{G}\hat{f}$ applied to this sub-groupoid has image in $\tilde{\mathcal{E}}^{[\gamma]} \mathbf{B} \subset (\mathcal{GB})^n$, and defines our map $\tilde{\mathbf{ev}}^{[\gamma]}$.

$$\tilde{\mathbf{ev}}^{[\gamma]}: \tilde{\mathbf{F}}(\hat{f}) \rightarrow \tilde{\mathcal{E}}^{[\gamma]} \mathbf{B}$$

The map $\tilde{\mathbf{ev}}_0$ we have seen before. After noting that $\tilde{\mathcal{E}}_0^{[\gamma]}(\mathbf{B}) = \prod_e \bar{\mathbf{B}}_e$, $\tilde{\mathbf{ev}}_0$ is equal to $\prod_e ev_e: \mathbf{F}(\hat{f}^+) \rightarrow \prod_e \bar{\mathbf{B}}_e$, where ev_e first appears in Definition 2.2, and stars in Theorem 2.5. As the quotient groupoid of $\tilde{\mathbf{F}}(\hat{f})$ is $\mathbf{F}(\hat{f})$, the corresponding map on quotient groupoids defines an evaluation map $\mathbf{ev}^{[\gamma]}$.

$$\mathbf{ev}^{[\gamma]}: \mathbf{F}(\hat{f}) \rightarrow \mathcal{E}^{[\gamma]} \mathbf{B}$$

As with $\tilde{\mathbf{ev}}$, we may consider $\tilde{\mathbf{ev}}_i^{[\gamma]}$ as giving maps from moduli stacks as in the following commutative diagram.

$$\begin{array}{ccccc} \tilde{\mathbf{F}}_1(\hat{f}) & \longrightarrow & \mathcal{M}_{[\gamma]++}^\bullet & \xrightarrow{\tilde{\mathbf{ev}}_1^{[\gamma]}} & \tilde{\mathcal{E}}_1^{[\gamma]} \mathbf{B} \\ \Downarrow & & \Downarrow & & \Downarrow \\ \tilde{\mathbf{F}}_0(\hat{f}) & \longrightarrow & \mathcal{M}_{[\gamma]+}^\bullet & \xrightarrow{\tilde{\mathbf{ev}}_0^{[\gamma]}} & \tilde{\mathcal{E}}_0^{[\gamma]} \mathbf{B} \\ \downarrow & & \downarrow & & \downarrow \\ \mathbf{F}(\hat{f}) & \longrightarrow & \mathcal{M}_{[\gamma]}^\bullet & \xrightarrow{\mathbf{ev}^{[\gamma]}} & \mathcal{E}^{[\gamma]} \mathbf{B} \longrightarrow \text{End}^{[\gamma]} \mathbf{B} \\ & & & \searrow \text{ev}^{[\gamma]} \nearrow & \end{array}$$

In the above, $\mathcal{M}_{[\gamma]++}^\bullet := \mathcal{M}_{[\gamma]+}^\bullet \times_{\mathcal{M}_{[\gamma]}^\bullet} \mathcal{M}_{[\gamma]+}^\bullet$ is the moduli stack of curves in $\mathcal{M}_{[\gamma]}^\bullet$ with two chosen points on each edge labeled by an edge of γ . We may use $\tilde{\mathbf{ev}}^{[\gamma]}: \mathcal{M}_{[\gamma]}^\bullet \rightarrow \tilde{\mathcal{E}}^{[\gamma]}(\mathbf{B})$ as a shorthand for the top-right square of the above diagram.

All curves in $\mathcal{M}_{[\gamma_v^+]}^\bullet(\mathbf{B})$ have image contained in $\bar{\mathbf{B}}_v$, so the inclusion $\bar{\mathbf{B}}_v \subset \mathbf{B}|_v$ induces an inclusion $\mathcal{M}_{[\gamma_v^+]}^\bullet(\mathbf{B}) \subset \mathcal{M}_{[\gamma_v^+]}^\bullet(\mathbf{B}|_v)$. Theorem 2.5 along with the observation that γ_v^+ has no internal edges and has all external edges infinitely extendable, implies that the cutting map applied to $\mathcal{M}_{[\gamma_v^+]}^\bullet(\mathbf{B}|_v)$ is an isomorphism, so

$$\mathcal{M}_{[\gamma_v^+]}^\bullet(\mathbf{B}|_v) = \mathcal{M}_{[\gamma_v^+]}^\bullet(\mathbf{B}|_v) .$$

Define cut_0 to be the following composition.

$$\begin{array}{c} \mathcal{M}_{[\gamma]}^\bullet(\mathbf{B}) \xrightarrow{\text{cut}} \prod_v \mathcal{M}_{[\gamma_v^+]}^\bullet(\mathbf{B}) \xrightarrow{\text{cut}_0} \prod_v \mathcal{M}_{[\gamma_v^+]}^\bullet(\mathbf{B}|_v) \end{array}$$

Our map cut_0 determines maps, cut_1 and cut , so that the following diagram commutes.

$$(5) \quad \begin{array}{ccccc} \mathcal{M}_{[\gamma]^{++}}^\bullet(\mathbf{B}) & \xrightleftharpoons[t]{s} & \mathcal{M}_{[\gamma]^+}^\bullet(\mathbf{B}) & \xrightarrow{\pi} & \mathcal{M}_{[\gamma]}^\bullet(\mathbf{B}) \\ \downarrow \text{cut}_1 & & \downarrow \text{cut}_0 & & \downarrow \text{cut} \\ \prod_v \mathcal{M}_{[\gamma_v]^{++}}^\bullet(\mathbf{B}|_v) & \xrightleftharpoons[t]{s} & \prod_v \mathcal{M}_{[\gamma_v]^+}^\bullet(\mathbf{B}|_v) & \xrightarrow{\pi} & \prod_v \mathcal{M}_{[\gamma_v]}^\bullet(\mathbf{B}|_v) \end{array}$$

So long as γ has internal edges, $(\text{cut } \hat{f})^+$ is larger than $\text{cut}_0(\hat{f}^+)$, because $\text{cut } \hat{f}$ has two edges for each internal edge of γ . The universal property of $(\text{cut } \hat{f})^+$ and $(\text{cut } \hat{f})^{++}$ give maps $\text{cut}_0(\hat{f}^+) \rightarrow (\text{cut}_0 \hat{f})^+$ and $\text{cut}_1(\hat{f}^{++}) \rightarrow (\text{cut } \hat{f})^{++}$ that define a map of groupoids $\tilde{\mathbf{F}}(\hat{f}) \rightarrow \tilde{\mathbf{F}}(\text{cut } \hat{f})$.

Use the notation $\mathcal{M}_{[\gamma_v]}^\bullet(\mathbf{B}) \subset \mathcal{M}_{[\gamma_v]}^\bullet(\mathbf{B}|_v)$ for the substack consisting of curves whose tropical part has all vertices in $\underline{\mathbf{B}}_v \subset \mathbf{B}|_v$. This is the image of $\mathcal{M}_{[\gamma_v]}^\bullet(\mathbf{B}) \subset \mathcal{M}_{[\gamma_v]^+}^\bullet(\mathbf{B}|_v)$ under the projection forgetting the extra points on edges. From Theorem 2.5, one could expect that the following diagram

$$\begin{array}{ccc} \mathcal{M}_{[\gamma]}^\bullet(\mathbf{B}) & \xrightarrow{\text{ev}^{[\gamma]}} & \mathcal{E}^{[\gamma]} \mathbf{B} \\ \downarrow \text{cut} & & \downarrow \Delta \\ \prod_v \mathcal{M}_{[\gamma_v]}^\bullet(\mathbf{B}) & \xrightarrow{\prod_v \text{ev}^{[\gamma_v]}} & \prod_v \mathcal{E}^{[\gamma_v]} \mathbf{B}|_v \end{array}$$

is close to being a fiber-product diagram. It fails to be a fiber-product diagram because, in taking the quotient, we have thrown away tropical information encoding the requirement that the length of an internal edge of a tropical curve is positive. We shall deal with this issue using tropical completion, but shall do so on the level of Kuranishi structures instead of using $\mathcal{M}_{[\gamma]}^\bullet$.

Proposition 3.6. *Given any family of curves \hat{f} in $\prod_v \mathcal{M}_{[\gamma_v]}^\bullet(\mathbf{B})$ so that $\prod_v \text{ev}^{[\gamma]}$ is transverse to Δ , there exists a family of curves, $\text{cut}^*(\hat{f})$, in $\mathcal{M}_{[\gamma]}^\bullet(\mathbf{B})$ with a map*

$$\text{cut}(\text{cut}^*(\hat{f})) \rightarrow \hat{f}$$

satisfying the following universal property: Given any family of curves \hat{h} in $\mathcal{M}_{[\gamma]}^\bullet(\mathbf{B})$ and a map $\psi: \text{cut}(\hat{h}) \rightarrow \hat{f}$, there exists a unique map $\text{cut}^(\psi): \hat{h} \rightarrow \text{cut}^*(\hat{f})$ so that the following diagram commutes.*

$$(6) \quad \begin{array}{ccc} & \text{cut}(\text{cut}^*(\hat{f})) & \\ \text{cut}(\text{cut}^*(\psi)) \nearrow & \downarrow & \\ \text{cut}(\hat{h}) & \xrightarrow{\psi} & \hat{f} \end{array}$$

Moreover, the following diagram

$$(7) \quad \begin{array}{ccc} \tilde{\mathbf{F}}(\text{cut}^*(\hat{f})) & \xrightarrow{\tilde{\text{ev}}^{[\gamma]}} & \tilde{\mathcal{E}}^{[\gamma]}(\mathbf{B}) \\ \downarrow & & \downarrow \tilde{\Delta} \\ \tilde{\mathbf{F}}(\hat{f}) & \xrightarrow{\prod_v \tilde{\text{ev}}^{[\gamma_v]}} & \prod_v \tilde{\mathcal{E}}^{[\gamma_v]}(\mathbf{B}|_v) \end{array}$$

is a fiber-product diagram of groupoids.

Proof:

Theorem 2.5 allows us to construct $\text{cut}_0^*(\hat{f}^+)$ so that the following is a fiber-product diagram.

$$\begin{array}{ccc} \mathbf{F}(\text{cut}_0^* \hat{f}^+) & \xrightarrow{\tilde{\mathbf{e}}\tilde{\mathbf{v}}_0^{[\gamma]}} & \tilde{\mathcal{E}}_0^{[\gamma]}(\mathbf{B}) \\ \downarrow & & \downarrow \tilde{\Delta}_0 \\ \mathbf{F}(\hat{f}^+) & \xrightarrow{\prod_v \tilde{\mathbf{e}}\tilde{\mathbf{v}}_0^{[\gamma_v]}} & \prod_v \tilde{\mathcal{E}}^{[\gamma_v]}(\mathbf{B}|_v) \end{array}$$

In particular, define $\text{cut}_0^*(\hat{f}^+)$ as the construction, cut^* , from Theorem 2.5 applied to the fiber-product of \hat{f}^+ with $\tilde{\mathcal{E}}_0^{[\gamma]}(\mathbf{B})$: taking this fiber-product gives a family of curves in $\mathcal{M}_{[\gamma_v]^+}^\bullet(\mathbf{B}|_v)$ which, when cut, are contained entirely inside $\tilde{\mathbf{B}}_v \subset \mathbf{B}|_v$, so correspond to a family of curves in $\mathcal{M}_{[\gamma_v]}^\bullet(\mathbf{B})$ to which Theorem 2.5 applies. The fiber-product exists because $\prod_v \tilde{\mathbf{e}}\tilde{\mathbf{v}}_0^{[\gamma_v]}$ being transverse to Δ implies that $\prod_v \tilde{\mathbf{e}}\tilde{\mathbf{v}}_0^{[\gamma_v]}$ is transverse to $\tilde{\Delta}_0$.

The universal property from Theorem 2.5 implies the corresponding universal property for $\text{cut}_0^* \hat{f}^+$: given any family of curves \hat{h} in $\mathcal{M}_{[\gamma]^+}^\bullet(\mathbf{B})$, any map $\psi: \text{cut}_0 \hat{h} \rightarrow \hat{f}^+$ has a unique pullback, $\text{cut}_0^* \psi: \hat{h} \rightarrow \text{cut}_0^*(\hat{f}^+)$ so that the following diagram commutes.

$$\begin{array}{ccc} & \text{cut}_0(\text{cut}_0^*(\hat{f}^+)) & \\ \text{cut}_0(\text{cut}_0^*(\psi)) \nearrow & \downarrow & \\ \text{cut}(\hat{h}) \xrightarrow{\psi} & \hat{f}^+ & \end{array}$$

We shall construct $\text{cut}^* \hat{f}$ so that $\text{cut}_0^*(\hat{f}^+) = (\text{cut} \hat{f})^+$ (or rather these two families are canonically isomorphic, because of the universal property of pullbacks and the commutative diagram (5)). Using the notation of diagram (5), $\pi \circ \text{cut}_0 \circ s = \pi \circ \text{cut}_0 \circ t = \pi \circ s \circ \text{cut}_1$, so there are canonical isomorphisms (written as equalities) as follows:

$$s^* \text{cut}_0^* \hat{f}^+ = t^* \text{cut}_0^* \hat{f}^+ := \text{cut}_1^*(\hat{f}^{++})$$

The natural maps

$$ss^* \text{cut}_0^* \hat{f}^+ \rightarrow \text{cut}_0^* \hat{f}^+$$

$$tt^* \text{cut}_0^* \hat{f}^+ \rightarrow \text{cut}_0^* \hat{f}^+$$

project to give the following groupoid internal to $\mathcal{M}_{[\gamma]}^\bullet(\mathbf{B})$.

$$\pi s \text{cut}_1^*(\hat{f}^{++}) \rightrightarrows \pi \text{cut}_0^*(\hat{f}^+)$$

Define $\text{cut}^* \hat{f}$ to be the quotient of this groupoid. (The groupoid action is free, so $\mathbf{F}(\text{cut}^* \hat{f})$ is an exploded manifold instead of a stack.) The map $\pi \text{cut}_0^* \hat{f}^+ \rightarrow \text{cut}^* \hat{f}$ lifts uniquely to a map $\text{cut}_0^* \hat{f}^+ \rightarrow (\text{cut}^* \hat{f})^+$ with inverse the unique lift of the map $\pi \circ \text{cut}_0(\text{cut}^* \hat{f})^+ \rightarrow \text{cut}^* \hat{f}$ using the universal property of cut_0^* and π^* . So $(\text{cut}^* \hat{f})^+$ is indeed canonically isomorphic to $\text{cut}_0^* \hat{f}^+$.

The universal property for $\text{cut}^* \hat{f}$ is implied by the universal property for $\text{cut}_0^* \hat{f}^+$. In particular, given any \hat{h} in $\mathcal{M}_{[\gamma]^+}^\bullet(\mathbf{B})$, any map $\psi: \text{cut} \pi \hat{h} \rightarrow \text{cut}^* \hat{f}$ has a unique lift to $\psi^+: \text{cut}_0 \hat{h} \rightarrow \hat{f}^+$ and $\text{cut}_0^* \psi^+: \hat{h} \rightarrow \text{cut}_0^* \hat{f}^+ = (\text{cut}^* \hat{f})^+$. Projecting this map using π gives the required unique lift $\text{cut}^* \psi: \pi \hat{h} \rightarrow \text{cut}^* \hat{f}$. As all families in $\mathcal{M}_{[\gamma]}^\bullet(\mathbf{B})$ are locally in the image of π , the universal property of cut^* holds for all families.

It remains to show that diagram (7) is a fiber-product diagram. We have already identified $\mathbf{F}(\text{cut}_0^* \hat{f}^+) = \tilde{\mathbf{F}}_0(\text{cut}^* \hat{f})$ as the appropriate fiber-product, so it remains to check that

$$(8) \quad \begin{array}{ccc} \tilde{\mathbf{F}}_1(\text{cut}^* \hat{f}) & \xrightarrow{\tilde{\mathbf{c}}\mathbf{v}_1^{[\gamma]}} & \tilde{\mathcal{E}}_1^{[\gamma]}(\mathbf{B}) \\ \downarrow & & \downarrow \tilde{\Delta} \\ \tilde{\mathbf{F}}_1(\hat{f}) & \xrightarrow{\prod_v \tilde{\mathbf{c}}\mathbf{v}_1^{[\gamma_v]}} & \prod_v \tilde{\mathcal{E}}_1^{[\gamma]}(\mathbf{B}|_v) \end{array}$$

is a fiber-product diagram. A map $x: \mathbf{A} \rightarrow \tilde{\mathbf{F}}_1(\text{cut}^* \hat{f})$ is equivalent to a map $x_0: \mathbf{A} \rightarrow \tilde{\mathbf{F}}_0(\text{cut}^* \hat{f})$ and for each edge e of γ , a map $c_e: \mathbf{A} \rightarrow \mathbf{T}$ so that the action of c_e on $x_0^*(\text{cut}^* \hat{f})^+$ is well-defined. We already know that x_0 is equivalent to the corresponding maps to $\tilde{\mathcal{E}}_0(\mathbf{B})$ and $\tilde{\mathbf{F}}_0(\hat{f})$ compatible with the fiber-product. Promoting these to maps to compatible maps to $\tilde{\mathcal{E}}_1(\mathbf{B})$ and $\tilde{\mathbf{F}}_1(\hat{f})$ is equivalent to choosing functions $c_e: \mathbf{A} \rightarrow \mathbf{T}$ for every edge of γ and $\prod_v \gamma_v$ so that

- (1) $c_{e_1} = c_{e_2}^{-1} = c_e$ if e is an internal edge of γ that breaks into e_1 and e_2 .
- (2) $c_e = c_{e'}$ if e' is the edge of $\prod_v \gamma_v$ corresponding to an end, e , of γ .
- (3) For edges e of $\prod_v \gamma_v$, the action of c_e on the pullback of \hat{f}^+ is well-defined.
- (4) For edges e of γ , the action of c_e on the image of \mathbf{A} in $\tilde{\mathcal{E}}_0^{[\gamma]} \mathbf{B}$ is well-defined.

These conditions on our functions are equivalent to a choice of c_e for each edge of γ so that the action on $x_0^*(\text{cut}^* \hat{f})^+$ is well-defined: Conditions (1) and (2) ensure that we only need c_e for edges of γ ; condition (3) (along with the identifications from the previous conditions) is equivalent to the action of c_e being well-defined on $x_0^*(\text{cut}^* \hat{f})^+$, and condition (4) follows from the previous conditions and the fact that the maps to $\tilde{\mathcal{E}}_0(\mathbf{B})$ and $\tilde{\mathbf{F}}_0(\hat{f})$ are compatible. Therefore, diagram (8) is a fiber-product diagram, and the proof that diagram (7) is a fiber-product diagram is complete. \square

4. SIMPLE GLUING FORMULA

Proposition 3.6 implies that the following is almost a fiber-product diagram

$$\begin{array}{ccc} \mathcal{M}_{[\gamma]}^{st}(\mathbf{B}) & \xrightarrow{\mathbf{c}\mathbf{v}^{[\gamma]}} & \mathcal{E}^{[\gamma]}(\mathbf{B}) \\ \downarrow \text{cut} & & \downarrow \Delta \\ \prod_v \mathcal{M}_{[\gamma_v]}^{st}(\mathbf{B}|_v) & \xrightarrow{\prod_v \mathbf{c}\mathbf{v}^{[\gamma_v]}} & \prod_v \mathcal{E}^{[\gamma_v]}(\mathbf{B}|_v) \end{array}$$

so we could hope that a version of the usual relationship for pushforwards of differential forms holds. It does hold after suitable tropical completion, and results in our gluing formula. Of course, $\mathbf{c}\mathbf{v}_i^{[\gamma]}$, as constructed in [25], involves pushing forward from the virtual fundamental class, which is the intersection with 0 of a section $\bar{\partial}$ of a sheaf \mathcal{Y} over \mathcal{M}^{st} . To prove our gluing formula, we must first verify that our diagram above is compatible with $\bar{\partial}$ and \mathcal{Y} .

We now describe a natural identification of \mathcal{Y} over $\mathcal{M}_{[\gamma]}^{st}(\mathbf{B})$ with $\text{cut}^* \mathcal{Y}$. Using the cutting map, a family of curves \hat{f} is sent to several families of curves, \hat{f}_v , all parametrized by $\mathbf{F}(\hat{f})$. There is an identification, of sheaves of $C^{\infty,1}(\mathbf{F}(\hat{f}))$ -modules, in the following form.

$$\mathcal{Y}(\hat{f}) \cong \oplus_v \mathcal{Y}(\hat{f}_v)$$

Using the cutting map, the strata, $\mathbf{C}(\hat{f})_v$, of $\mathbf{C}(\hat{f})$ labelled by v have a natural inclusion into $\mathbf{C}(\hat{f}_v)$, compatible with the inclusion, $\bar{\mathbf{B}}_v \subset \mathbf{B}|_v$, of the closure of the

stratum of \mathbf{B} containing v into $\mathbf{B}|_v$.

$$(9) \quad \begin{array}{ccc} \mathbf{C}(\hat{f})_v & \xrightarrow{\hat{f}} & \bar{\mathbf{B}}_v \\ \downarrow & & \downarrow \\ \mathbf{C}(\hat{f}_v) & \xrightarrow{\hat{f}_v} & \mathbf{B}|_v \end{array}$$

The sheaf $\mathcal{Y}(\hat{f}_v)$ consists of $C^{\infty,1}$ sections of $T_{vert}^* \mathbf{C}(\hat{f}_v) \otimes \hat{f}_v^* T\mathbf{B}|_v$ that vanish on the edges of curves. Our inclusions above identify such sections with sections of $T_{vert}^* \mathbf{C}(\hat{f}) \otimes \hat{f}^* T\mathbf{B}$ over $\mathbf{C}(\hat{f})_v$. We can extend these sections to be 0 everywhere else on $\mathbf{C}(\hat{f})$; these extended sections are $C^{\infty,1}$ because our sections vanish on edges. This identification defines an inclusion $\mathcal{Y}(\hat{f}_v) \subset \mathcal{Y}(\hat{f})$.

The resulting map $\oplus_v \mathcal{Y}(\hat{f}_v) \rightarrow \mathcal{Y}(\hat{f})$ has an obvious inverse. Any section of $\mathcal{Y}(\hat{f})$ restricted to $\mathbf{C}(\hat{f})_v$, then extended as 0 elsewhere on $\mathbf{C}(\hat{f})$ defines a section of $\mathcal{Y}(\hat{f}_v)$. This construction for all v provides our required inverse map, $\mathcal{Y}(\hat{f}) \rightarrow \oplus_v \mathcal{Y}(\hat{f}_v)$. These identifications are clearly compatible with the $C^{\infty,1}(\mathbf{F}(\hat{f}))$ -module structure and the section $\bar{\partial}$, and are natural in the sense that given any map $\hat{f} \rightarrow \hat{g}$ in $\mathcal{M}_{[\gamma]}^{st}(\mathbf{B})$, the corresponding diagram commutes.

$$\begin{array}{ccc} \mathcal{Y}(\hat{f}) & \longleftarrow & \mathcal{Y}(\hat{g}) \\ \downarrow & & \downarrow \\ \oplus_v \mathcal{Y}(\hat{f}) & \longleftarrow & \oplus_v \mathcal{Y}(\hat{g}_v) \end{array}$$

This completes the description of our natural identification

$$\mathcal{Y}(\mathcal{M}_{[\gamma]}^{st}(\mathbf{B})) = \text{cut}^*(\mathcal{Y}(\prod_v \mathcal{M}_{[\gamma_v]}^{st}(\mathbf{B}|_v))) .$$

The virtual fundamental class of the moduli stack of holomorphic curves, constructed in [25], uses an embedded Kuranishi structure, constructed in [23]. Each Kuranishi chart on $\mathcal{M}_{\bullet}^{st}$ is an open substack $\mathcal{U} \subset \mathcal{M}_{\bullet}^{st}$, and on \mathcal{U} a finite-dimensional obstruction bundle $V \subset \mathcal{Y}$, and a family \hat{f} with automorphism group G so that \hat{f}/G represents $\bar{\partial}^{-1}(V) \subset \mathcal{U}$.

Remark 4.1. *Proposition 3.6 implies that if \hat{f}/G represents the substack $\bar{\partial}^{-1}(V)$, then $(\text{cut}^* \hat{f})/G$ represents the substack $\bar{\partial}^{-1}(\text{cut}^* V)$.*

The pullback of a Kuranishi chart $(\mathcal{U}, V, \hat{f}/G)$ on $\prod_v \mathcal{M}_{[\gamma_v]}^{st}(\mathbf{B}_v)$ is

$$\text{cut}^*(\mathcal{U}, V, \hat{f}/G) := (\text{cut}^{-1} \mathcal{U}, \text{cut}^* V, (\text{cut}^* \hat{f})/G)$$

however we need to check that our pulled-back Kuranishi chart satisfies the technical conditions of Definition 2.27 of [23]. To be used for a Kuranishi chart as defined in [23], an obstruction bundle must be simply-generated in the sense of Definition 2.24 of [23].

Lemma 4.2. *Let $f \in \mathcal{M}_{[\gamma]}^{st}(\mathbf{B})$ be a holomorphic curve with $\text{cut } f = \prod_v f_v$. Given any choice of simply-generated obstruction bundles V_v , defined on a open neighborhoods of f_v in $\mathcal{M}_{[\gamma_v]}^{st}(\mathbf{B}|_v)$, the pullback, $\text{cut}^*(\oplus V_v)$, is simply-generated on an open neighborhood of f in $\mathcal{M}_{[\gamma]}^{st}(\mathbf{B})$.*

Proof:

Lemma 6.4 from [23] implies that, on a small enough neighborhood, \mathcal{U}_v of f_v , V_v is pulled back (Definition 2.23 of [23]) from some family of curves \hat{g}_v in \mathcal{U}_v with a

group G_v of automorphisms using a map

$$\begin{array}{ccc} \mathcal{U}_v^{+1} & \longrightarrow & \mathbf{C}(\hat{g}_v)/G_v \\ \downarrow & & \downarrow \\ \mathcal{U}_v & \longrightarrow & \mathbf{F}(\hat{g}_v)/G_v \end{array}$$

that is a holomorphic isomorphism restricted to each fiber. Consider $\prod_v \hat{g}_v$ as a family of (possibly unstable) holomorphic curves, mapping to a point, and decorated by γ_v . Let \hat{g} be $\text{cut}^*(\prod_v \hat{g}_v)$ in $\mathcal{M}_{[\gamma]}^{\infty,1}(pt)$. As specified by Proposition 3.6, this \hat{g} comes with a natural morphism $\text{cut } \hat{g} \rightarrow \prod_v \hat{g}_v$, which satisfies the property that given any family of curves \hat{f} in $\mathcal{M}_{[\gamma]}^{\infty,1}(pt)$ with a map $\psi: \text{cut}(\hat{f}) \rightarrow \prod_v \hat{g}_v$, there exists a unique map $\psi': \hat{f} \rightarrow \hat{g}$ so that the following diagram commutes:

$$\begin{array}{ccc} & \psi & \\ \text{cut}(\hat{f}) & \xrightarrow{\text{cut}(\psi')} & \text{cut } \hat{g} \longrightarrow \prod_v g_v \end{array}$$

This universal property implies that the fiberwise holomorphic map

$$\begin{array}{ccc} \prod_v \mathcal{U}_v^{+1} & \longrightarrow & \prod_v \mathbf{C}(\hat{g}_v)/G_v \\ \downarrow & & \downarrow \\ \prod_v \mathcal{U}_v & \longrightarrow & \prod_v \mathbf{F}(\hat{g}_v)/G_v \end{array}$$

lifts canonically to a fiberwise holomorphic map

$$(10) \quad \begin{array}{ccc} \mathcal{U}^{+1} & \longrightarrow & \mathbf{C}(\hat{g})/\prod_v G_v \\ \downarrow & & \downarrow \\ \mathcal{U} & \longrightarrow & \mathbf{F}(\hat{g})/\prod_v G_v \end{array}$$

where \mathcal{U} is the inverse image of $\prod_v \mathcal{U}_v$ under the cutting map. We shall now show that using the above map (10), $\text{cut}^*(\oplus_v V_v)$ is the pullback of a sheaf of sections in the sense of Definition 2.23 of [23], and in particular $\text{cut}^*(\oplus_v V_v)$ is simply-generated in the sense of Definition 2.24. As stipulated by these definitions, V_v is constructed using the pullback of some sections σ of $\Gamma^{0,1}(T_{vert}^* \mathbf{C}(\hat{g}_v) \otimes T(\mathbf{B}_{[v]}))$ vanishing on edges of $\mathbf{C}(\hat{g}_v)$. Using the inclusions (9), and the map $\mathbf{C}(\text{cut } \hat{g}) \rightarrow \mathbf{C}(\prod_v \hat{g}_v)$, any such section σ can be pulled back, then extended to a section σ' of $\Gamma^{0,1}(T_{vert}^* \mathbf{C}(\hat{g}) \otimes T\mathbf{B})$, vanishing outside the strata of $\mathbf{C}(\hat{g})$ labeled by v . These sections σ' , when pulled back²¹ using our map (10) generate $\text{cut}^*(\oplus_v V)$ in the sense of 2.24 of [23].

This completes the proof that the pullback of $\oplus_v V_v$ is simply-generated on \mathcal{U} . \square

Proposition 4.3. *Given any choice of $ev^{[\gamma_v]}$ -submersive embedded Kuranishi structure on $\mathcal{M}_{[\gamma_v]}^{st}(\mathbf{B}_{[v]}^\vee)$, there exists an embedded Kuranishi structure on $\prod_v \mathcal{M}_{[\gamma_v]}^{st}(\mathbf{B}_{[v]}^\vee)$ so that every Kuranishi chart has an extension that is a product of charts from the chosen Kuranishi structures on $\mathcal{M}_{[\gamma_v]}^{st}(\mathbf{B}_{[v]}^\vee)$, and so that the pullback of this Kuranishi structure to $\mathcal{M}_{[\gamma]}^{st}(\mathbf{B})$ is an embedded Kuranishi structure.*

²¹Definition 2.23 of [23].

Proof: Even though the Kuranishi charts on $\mathcal{M}_{[\gamma_v]}^{st}(\mathbf{B}|_v)$ are compatible with each other, the product Kuranishi charts on $\prod_v \mathcal{M}_{[\gamma_v]}^{st}(\mathbf{B}|_v)$ and the pulled-back Kuranishi charts on $\mathcal{M}_{[\gamma]}^{st}(\mathbf{B})$ may not be compatible (because the product of totally ordered sets usually only has a partial order.) To overcome this problem, we choose where to use each chart carefully. The construction is analogous to the construction of compatible Kuranishi charts in the proof of Theorem 7.3 from [23].

In particular, Lemma 7.1 of [23] and Lemma 4.2 above, imply that each holomorphic curve f in $\prod_v \mathcal{M}_{[\gamma_v]}^{st}$ has a neighborhood \mathcal{O} with a $C^{\infty,1}$ function $\rho: \mathcal{O} \rightarrow [0, 1]$ so that

- all holomorphic curves in the closure of $\{\rho > 0\} \subset \prod_v \mathcal{M}_{[\gamma_v]}^{st}$ are contained in \mathcal{O} ;
- $\rho(f) = 1$
- there exists some collection $(\mathcal{U}_v, V_v, \hat{f}_v/G_v)$ of the given Kuranishi charts on $\mathcal{M}_{[\gamma_v]}^{st}$ so that $\mathcal{O} \subset \prod_v \mathcal{U}_v$;
- $\text{cut}^* \oplus_v V_v$ is simply-generated on $\text{cut}^{-1}(\mathcal{O})$.

As the set of holomorphic curves in any connected component of $\prod_v \mathcal{M}_{[\gamma_v]}^{st}(\mathbf{B}|_v)$ is compact²², there exists some finite collection $\{(\mathcal{O}_1, \rho_1), \dots, (\mathcal{O}_N, \rho_N)\}$ with the substacks $\{\rho_i > \frac{1}{2}\}$ covering the holomorphic curves in this component of $\prod_v \mathcal{M}_{[\gamma_v]}^{st}(\mathbf{B}|_v)$. Use the notation V_i for the obstruction bundle, $\oplus_v V_v$ above, defined on \mathcal{O}_i , and for any $S \subset \{1, \dots, N\}$, let $V_S := \oplus_{i \in S} V_i$.

We shall use V_S on a restricted domain, \mathcal{O}_S , defined to be the interior of the following set.

$$\left\{ \left(\min(0.4, \min_{j \in S} \rho_j) - \max(0.1, \max_{j' \notin S} \rho_{j'}) \right) > \frac{0.1}{N} \right\} \subset \bigcap_{j \in S} \mathcal{O}_j$$

In the above, we extend $\rho_{j'}$ to be 0 wherever it is not already defined. The proof of Claim 7.4 from [23] applies without modification to show that that these \mathcal{O}_S form an open cover of the holomorphic curves in our connected component of $\prod_v \mathcal{M}_{[\gamma_v]}^{st}(\mathbf{B}|_v)$. Make a similar construction for all other connected components.

By construction, $\mathcal{O}_S \cap \mathcal{O}_{S'} \neq \emptyset$ only if V_S is a subsheaf of $V_{S'}$ or visa versa, so we may construct compatible Kuranishi charts on $\prod_v \mathcal{M}_{[\gamma_v]}^{st}$ using subcharts of the product chart that uses V_S only within \mathcal{O}_S .

Let us verify that each of our Kuranishi charts, $(\mathcal{O}, V, \hat{f}/G)$ on $\prod_v \mathcal{M}_{[\gamma_v]}^{st}(\mathbf{B}|_v)$, pulls back to a Kuranishi chart $(\text{cut}^{-1} \mathcal{O}, \text{cut}^* V, \text{cut}^* \hat{f}/G)$ on $\mathcal{M}_{[\gamma]}^{st}(\mathbf{B})$. Our assumption that our original charts were $ev^{[\gamma_v]}$ -submersive implies that our chart is $(\prod_v ev^{[\gamma_v]})$ -submersive, and in particular, $\prod_v ev^{[\gamma_v]}$ restricted to \hat{f} is a submersion. Proposition 3.6 then tells us that $\text{cut}^* \hat{f}$ is a well-defined $C^{\infty,1}$ family of curves, and that $\text{cut}^* \hat{f}/G$ represents $\bar{\partial}^{-1}(\text{cut}^* V) \subset \text{cut}^{-1}(\mathcal{O})$. We also need that $D\bar{\partial}$ is strongly transverse²³ to $\text{cut}^* V$ at any holomorphic curve f in \hat{f} . By our submersive assumption, $D\bar{\partial}$ at $\text{cut} f$ is strongly transverse to V , even when restricted to the kernel of the derivative of $\prod_v ev^{[\gamma_v]}$. Proposition 3.6 then implies that $D\bar{\partial}$ is strongly transverse to $\text{cut}^* V$. We already know that $\text{cut}^* V$ is simply-generated, and $\text{cut}^* V$ is complex because V is, so $(\text{cut}^{-1} \mathcal{O}, \text{cut}^* V, \text{cut}^* \hat{f}/G)$ satisfies all the requirements to be a Kuranishi chart from Definition 2.27 of [23].

²²To ensure compactness, we have assumed that there is a taming form taming the almost complex structure on \mathbf{B} , and that \mathbf{B} admits an immersion into some \mathbb{R}^N . These conditions, and the required compactness, then automatically hold for $\mathbf{B}|_v$.

²³See definitions 2.26 and 2.29 of [23].

The pullback of any compatible collection of Kuranishi charts is compatible, so our embedded Kuranishi structure on $\prod_v \mathcal{M}_{[\gamma_v]}^{st}(\mathbf{B}|_v)$ pulls back to an embedded Kuranishi structure on $\mathcal{M}_{[\gamma]}^{st}(\mathbf{B})$. \square

Proposition 3.6 almost describes $\text{cut}^* \hat{f}$ as a fiber-product. The following lemma proves that after applying tropical completion as in section 7 of [25], we get an honest fiber-product.

Lemma 4.4. *Given any Kuranishi chart $(\mathcal{U}, V, \hat{f}/G)$ on $\prod_v \mathcal{M}_{[\gamma_v]}^{st}(\mathbf{B}|_v)$ that pulls back to a Kuranishi chart $(\text{cut}^{-1} U, \text{cut}^* V, \text{cut}^* \hat{f}/G)$, the following is a fiber-product diagram.*

$$\begin{array}{ccc} \mathbf{F}(\text{cut}^* \hat{f})|_{\gamma} & \xrightarrow{\text{ev}^{[\gamma]}|_{\gamma}} & (\mathcal{E}^{[\gamma]} \mathbf{B})|_{\text{ev}^{[\gamma]} \gamma} \\ \downarrow & & \downarrow \Delta \\ \mathbf{F}(\hat{f}) & \xrightarrow{\prod_v \text{ev}^{[\gamma_v]}} & \prod_v \mathcal{E}^{[\gamma_v]}(\mathbf{B}|_v) \end{array}$$

As first glance, the above diagram requires tropical completion of the bottom row at $\prod_v \gamma_v$ to make sense, however as we shall see in the proof, such tropical completion does nothing.

Proof:

Let us describe the fiber-product $\mathbf{F}(\hat{f}) \times_{\prod_v \mathcal{E}^{[\gamma_v]}(\mathbf{B}|_v)} \mathcal{E}^{[\gamma]}(\mathbf{B})$. The image of $\prod_v \text{ev}^{[\gamma_v]}$ is encoded by the following diagram

$$\begin{array}{ccccc} & & \prod_v \text{ev}_0^{[\gamma]} & & \\ & \swarrow & & \searrow & \\ \mathbf{F}(\hat{f}^+) & \longleftrightarrow & \tilde{E}_0 & \longrightarrow & \prod_v \tilde{\mathcal{E}}_0^{[\gamma_v]}(\mathbf{B}|_v) \\ & \searrow & \downarrow & & \\ & & \mathbf{F}(\hat{f}) & & \end{array}$$

where \tilde{E}_0 is the unique extension of the bundle $\mathbf{F}(\hat{f}^+) \rightarrow \mathbf{F}(\hat{f})$ so that the following is a fiber-product diagram.

$$(11) \quad \begin{array}{ccc} \tilde{E}_1 := \tilde{E}_0 \times_{\mathbf{F}(\hat{f})} \tilde{E}_0 & \longrightarrow & \prod_v \tilde{\mathcal{E}}_1^{[\gamma_v]}(\mathbf{B}|_v) \\ \Downarrow & & \Downarrow \\ \tilde{E}_0 & \longrightarrow & \prod_v \tilde{\mathcal{E}}_0^{[\gamma_v]}(\mathbf{B}|_v) \end{array}$$

Importantly, $\mathbf{F}(\hat{f}^+)$ is a subset of \tilde{E}_0 determined by an open condition on the tropical part of \tilde{E}_0 .

Claim 4.5. *The fiber-product*

$$\mathbf{F}(\hat{f}) \times_{\prod_v \mathcal{E}^{[\gamma_v]}(\mathbf{B}|_v)} \mathcal{E}^{[\gamma]}(\mathbf{B})$$

is the quotient stack of the groupoid

$$\tilde{E} \times_{\prod_v \tilde{\mathcal{E}}^{[\gamma_v]}(\mathbf{B}|_v)} \tilde{\mathcal{E}}^{[\gamma]}(\mathbf{B}) .$$

To prove Claim 4.5, consider a map of an exploded manifold \mathbf{X} into the quotient stack of the above fiber-product of groupoids. Such a map is a bundle $\tilde{\mathbf{X}}_0 \rightarrow \mathbf{X}$

and a pullback diagram.

$$\begin{array}{ccc} \tilde{\mathbf{X}}_1 := \tilde{\mathbf{X}}_0 \times_{\mathbf{X}} \tilde{\mathbf{X}}_0 & \longrightarrow & \tilde{E}_1 \times_{\prod_v \tilde{\mathcal{E}}_1^{[\gamma_v]}(\mathbf{B})|_v} \tilde{\mathcal{E}}_1^{[\gamma]}(\mathbf{B}) \\ \Downarrow & & \Downarrow \\ \tilde{\mathbf{X}}_0 & \longrightarrow & \tilde{E}_0 \times_{\prod_v \tilde{\mathcal{E}}_0^{[\gamma_v]}(\mathbf{B})|_v} \tilde{\mathcal{E}}_0^{[\gamma]}(\mathbf{B}) \end{array}$$

Because of the pullback diagram (11), the above is a pullback diagram if and only if the induced diagram

$$(12) \quad \begin{array}{ccc} \tilde{\mathbf{X}}_1 & \longrightarrow & \tilde{\mathcal{E}}_1^{[\gamma]}(\mathbf{B}) \\ \Downarrow & & \Downarrow \\ \tilde{\mathbf{X}}_0 & \longrightarrow & \tilde{\mathcal{E}}_0^{[\gamma]}(\mathbf{B}) \end{array}$$

is a pullback diagram — such a diagram is a map of \mathbf{X} to $\mathcal{E}^{[\gamma]}(\mathbf{B})$.

A map of \mathbf{X} to our stack fiber-product is a map $h: \mathbf{X} \rightarrow \mathbf{F}(\hat{f})$, a diagram in the form (12), and an isomorphism between their images in $\prod_v \mathcal{E}^{[\gamma_v]}(\mathbf{B})|_v$. Such an isomorphism amounts to a lift of h to a groupoid map $\tilde{h}: \tilde{\mathbf{X}} \rightarrow \tilde{E}$ so that the following diagram commutes.

$$\begin{array}{ccc} \tilde{\mathbf{X}} & \longrightarrow & \tilde{\mathcal{E}}^{[\gamma]}(\mathbf{B}) \\ \downarrow \tilde{h} & & \downarrow \\ \tilde{E} & \longrightarrow & \prod_v \tilde{\mathcal{E}}^{[\gamma_v]}(\mathbf{B})|_v \end{array}$$

This data of our map of \mathbf{X} into the stack fiber-product is equivalent to a map of \mathbf{X} into the quotient stack of our fiber-product of groupoids, so Claim 4.5 is true.

We need to relate $\mathbf{F}(\text{cut}^* \hat{f})$ to this fiber-product. Proposition 3.6 gives $\tilde{\mathbf{F}}(\text{cut}^* \hat{f})$ as a fiber-product, however this fiber-product involves $\hat{f}' \subset \hat{f}$, the pullback of \hat{f} under the inclusion $\prod_v \mathcal{M}_{[\gamma_v]}^{st}(\mathbf{B}) \rightarrow \prod_v \mathcal{M}_{[\gamma_v]}^{st}(\mathbf{B})|_v$. This family \hat{f}' is the subfamily of \hat{f} consisting of curves with tropical parts having all vertices contained in $\underline{\mathbf{B}}_v \subset \mathbf{B}|_v$. In particular, $\mathbf{F}(\hat{f}') \subset \mathbf{F}(\hat{f})$ is a subset determined by restricting to the inverse image of an open²⁴ subset of $\underline{\mathbf{F}}(\hat{f})$. Therefore, Proposition 3.6 gives that $\mathbf{F}(\hat{f})$ represents the quotient stack of a subgroupoid of $\tilde{E} \times_{\prod_v \tilde{\mathcal{E}}^{[\gamma_v]}(\mathbf{B})|_v} \tilde{\mathcal{E}}^{[\gamma]}(\mathbf{B})$ determined by restricting to the inverse image of an open subset of its tropical part. It follows that $\mathbf{F}(\text{cut}^* \hat{f})$ represents a subset of the fiber-product

$$\mathbf{F}(\hat{f}) \times_{\prod_v \mathcal{E}^{[\gamma_v]}(\mathbf{B})|_v} \mathcal{E}^{[\gamma]}(\mathbf{B})$$

determined by restricting to the inverse image of an open subset of its tropical part. This subset includes all the points corresponding to curves with tropical part actually equal to $\coprod_v \gamma_v$, and points in $\mathcal{E}^{[\gamma]}(\mathbf{B})$ that are the image of curves with tropical part γ , so we may describe the subset, $\mathbf{F}(\text{cut}^* \hat{f})|_\gamma$, corresponding to curves with tropical part equal to γ as an honest fiber-product. Taking tropical completions²⁵

²⁴Note that open subsets of \mathbf{B} have closed image in $\underline{\mathbf{B}}$.

²⁵Section 7 of [25] only describes tropical completion of orbifolds, not general stacks, so we must specify what is meant by tropical completion of $\mathcal{E}^{[\gamma]} \mathbf{B}$. This stack is a global quotient of $\text{End}^{[\gamma]} \mathbf{B}$, so define its tropical completion as the corresponding quotient of the tropical completion of $\text{End}^{[\gamma]} \mathbf{B}$.

at the relevant points corresponding to γ gives a fiber-product diagram.

$$\begin{array}{ccc} \mathbf{F}(\text{cut}^* \hat{f})|_{\check{\gamma}} & \xrightarrow{\text{ev}^{[\gamma]}|_{\check{\gamma}}} & (\mathcal{E}^{[\gamma]} \mathbf{B})|_{\check{\text{ev}}^{[\gamma]} \gamma} \\ \downarrow & & \downarrow \Delta|_{\check{\text{ev}}^{[\gamma]} \gamma} \\ \mathbf{F}(\hat{f})|_{\coprod_v \gamma_v} & \xrightarrow{\prod_v \text{ev}^{[\gamma_v]}|_{\gamma_v}} & \prod_v (\mathcal{E}^{[\gamma_v]}(\mathbf{B}|_v))|_{\check{\text{ev}}^{[\gamma_v]} \gamma_v} \end{array}$$

The tropical completions on the bottom row do nothing, because the spaces involved are already complete, and have tropical parts which are always an infinite cone around the tropical completion point. This is because $\mathbf{B}|_v$ is an infinite cone around v . The rescaling action around v also acts on $\mathcal{E}^{[\gamma_v]}(\mathbf{B}|_v)$, preserving the image of curves with tropical part γ_v (and only this point), so tropical completion at this point does nothing. Similarly, because \hat{f} has universal tropical structure, the rescaling action also acts on $\mathbf{F}(\hat{f})$. This scaling action preserves only the point corresponding to curves with tropical part γ_v , so again tropical completion at this point does nothing.

Removing the unnecessary tropical completions from the bottom row gives the required fiber-product diagram. \square

We are now ready to write our first gluing theorem. Consider the map $ev: \mathcal{M}^{st}(\mathbf{B}) \rightarrow \coprod_n (\text{End } \mathbf{B})^n$, and define

$$\eta := ev_!(h^{2g-2+n} q^E)$$

where g , n and E are the locally constant functions recording the genus, number of ends, and ω -energy of curves, and h and q are dummy variables. In general, $ev^{[\gamma]}$ is not complete, even restricted to curves with bounded genus and energy, and $ev^{[\gamma]}$ may not be defined. Use tropical completion as in section 7 of [25] to define

$$\eta^{[\gamma]} := (ev^{[\gamma]}|_{\check{\gamma}})_!(h^{2g-2+n} q^E)$$

and similarly use tropical completion at γ to define

$$\eta|_{\check{\gamma}} := (ev|_{\check{\gamma}})_!(h^{2g-2+n} q^E).$$

Similarly, given any complex vector-bundle W over $\mathcal{M}^{st}(\mathbf{B})$, define

$$\eta(W) := ev_!(h^{2g-2+n} q^E c(W))$$

$$\eta^{[\gamma]}(W) := (ev^{[\gamma]}|_{\check{\gamma}})_!(h^{2g-2+n} q^E c(W|_{\check{\gamma}}))$$

$$\eta(W)|_{\check{\gamma}} := (ev|_{\check{\gamma}})_!(h^{2g-2+n} q^E c(W|_{\check{\gamma}}))$$

where $c(W)$ is the top Chern class of W .

Remark 4.6. In the case of γ_v , tropical completion at γ_v affects nothing, and $\eta^{[\gamma_v]}(W)$ is the restriction of $\eta(W)$ to $\text{End}^{[\gamma_v]}(\mathbf{B}|_v) \subset \coprod_n (\text{End } \mathbf{B}|_v)^n$.

Remark 4.7. Given any point $p \in \coprod_n (\text{End } \mathbf{B})^n$, say that $\gamma \in \underline{ev}^{-1}p$ if curves with tropical part γ are sent by ev to points with tropical part p . Lemma 7.7 of [25] gives that

$$\eta|_p = \sum_{\gamma \in \underline{ev}^{-1}p} \eta|_{\check{\gamma}}$$

and more generally,

$$\eta(W)|_p = \sum_{\gamma \in \underline{ev}^{-1}p} \eta(W)|_{\check{\gamma}}.$$

Note that the target of $ev^{[\gamma]}|_\gamma$ is not $\text{End}^{[\gamma]}(\mathbf{B})$, but its tropical completion at the image of γ , so $\eta^{[\gamma]}$ is a (refined²⁶) cohomology class on $\text{End}^{[\gamma]}(\mathbf{B})|_{\underline{ev^{[\gamma]}(\gamma)}}$. Indicate the tropical completion of Δ again by Δ .

$$\Delta: \text{End}^{[\gamma]}(\mathbf{B})|_{\underline{ev^{[\gamma]}(\gamma)}} \longrightarrow \prod_v \text{End}^{[\gamma_v]}(\mathbf{B}|_v)$$

Theorem 4.8.

$$\Delta^* \prod_v \eta^{[\gamma_v]} = k_\gamma \eta^{[\gamma]}$$

where $k_\gamma = \prod_{e \in \text{ied}\gamma} m_e$, the product of the multiplicities of the internal edges of γ . Similarly, given complex vector bundles W_v on $\mathcal{M}_{[\gamma_v]}^{st}$,

$$\Delta^* \prod_v \eta^{[\gamma_v]}(W_v) = k_\gamma \eta^{[\gamma]}(\text{cut}^* \oplus_v W_v)|_\gamma.$$

Proof:

Using [23], construct $ev^{[\gamma_v]}$ -submersive embedded Kuranishi structures on $\mathcal{M}_{[\gamma_v]}^{st}(\mathbf{B}|_v)$, then use Proposition 4.3 to construct a corresponding embedded Kuranishi structure on $\prod_v \mathcal{M}_{[\gamma_v]}^{st}(\mathbf{B}|_v)$. The resulting Kuranishi category \mathcal{K} within $\prod_v \mathcal{M}_{[\gamma_v]}^{st}(\mathbf{B}|_v)$ is a weak product of the Kuranishi categories \mathcal{K}_v from $\mathcal{M}_{[\gamma_v]}^{st}(\mathbf{B}|_v)$, so Theorem 6.2 from [25] gives the expected product relation when pushing forward using \mathcal{K} and $\prod_v ev^{[\gamma_v]}$, or \mathcal{K}_v and $ev^{[\gamma_v]}$.

$$(13) \quad \left(\prod_v ev^{[\gamma_v]} \right)_! \left(\prod_v \hbar^{2g_v-2+n_v} q^{E_v} c(W_v) \right) = \prod_v \eta^{[\gamma_v]}(W_v)$$

Now consider the Kuranishi category, $\text{cut}^* \mathcal{K}$, defined using the pullback of our embedded Kuranishi structure, and apply tropical completion²⁷ to $ev^{[\gamma]}$ considered as a map from $\text{cut}^* \mathcal{K}$. Lemma 4.4 implies that the following is a pullback diagram of Kuranishi categories.

$$\begin{array}{ccc} \text{cut}^* \mathcal{K}|_\gamma & \xrightarrow{ev^{[\gamma]}|_\gamma} & (\mathcal{E}^{[\gamma]}(\mathbf{B}))|_{\underline{ev^{[\gamma]}(\gamma)}} \\ \downarrow \text{cut}|_\gamma & & \downarrow \Delta \\ \mathcal{K} & \xrightarrow{\prod_v ev^{[\gamma_v]}} & \prod_v \mathcal{E}^{[\gamma_v]}(\mathbf{B}|_v) \end{array}$$

So long as the stacks above on the right are orbifolds, we can apply Theorem 5.22 of [25] to obtain the relationship between $(\mathbf{ev}^{[\gamma]}|_\gamma)_!$ and $(\prod_v \mathbf{ev}^{[\gamma_v]})_!$. We have that $\mathcal{E}^{[\gamma]}(\mathbf{B}) = \text{End}^{[\gamma]}(\mathbf{B}) / \prod_e G_e$, where the group G_e is \mathbb{Z}_{m_e} for each edge of multiplicity $m_e \neq 0$, and is \mathbf{T} for each edge of multiplicity 0. We can forget the G_e -action for each end e of γ , leaving us with the following pullback diagram.

$$\begin{array}{ccc} \text{cut}^* \mathcal{K}|_\gamma & \longrightarrow & (\text{End}^{[\gamma]}(\mathbf{B}))|_{\underline{ev^{[\gamma]}(\gamma)}} / \prod_{e \in \text{ied}\gamma} G_e \\ \downarrow \text{cut}|_\gamma & & \downarrow \\ \mathcal{K} & \longrightarrow & \prod_v \text{End}^{[\gamma_v]}(\mathbf{B}|_v) / \prod_{e \in \text{ied}\gamma} G_e^2 \end{array}$$

In particular, $\text{cut}^* \mathcal{K}|_\gamma$ is a $(\prod_{e \in \text{ied}\gamma} G_e)$ -bundle over the corresponding fiber-product forgetting the G_e -actions. If any internal edge has multiplicity 0, it follows that $\eta^{[\gamma]}$ must be 0. Otherwise, the righthand side of the above consists of orbifolds,

²⁶The minimal cohomology theory of exploded manifolds containing the usual cohomology but with pushforwards compatible with fiber-products is called refined cohomology. See section 9 of [19].

²⁷See section 7 of [25], especially Lemma 7.7.

so applying Theorem 5.22 of [25], then pushing forward the result via the map forgetting the G_e -actions, gives

$$(14) \quad (ev^{[\gamma]}|_{\gamma})_!(\text{cut}^* \theta) = \left(\prod_{e \in \text{ied} \gamma} m_e \right) \Delta^* \left(\prod_v ev^{[\gamma_v]} \right)_!(\theta)$$

Observing that $\text{cut}^* (\prod_v \hbar^{2g_v-2+n_v} q^{E_v} c(W_v)) = \hbar^{2g-2+n} q^E c(\text{cut}^* \oplus_v W_v)$, then combining equations (14), and (13) gives our desired result.

$$\Delta^* \prod_v \eta^{[\gamma_v]}(W_v) = k_{\gamma} \eta^{[\gamma]}(\text{cut}^* \oplus_v W_v)$$

□

Consider the map

$$i^{[\gamma]}: \text{End}^{[\gamma]}(\mathbf{B})|_{\underline{ev}[\gamma]\gamma} \longrightarrow \left(\prod_n (\text{End } \mathbf{B})^n \right)|_{\underline{ev}\gamma}$$

forgetting all internal edges.

Lemma 4.9. *The following relationship holds between $\eta|_{\gamma}$ and $\eta^{[\gamma]}$.*

$$\eta|_{\gamma} = \frac{1}{|\text{Aut } \gamma|} i^{[\gamma]}_! \eta^{[\gamma]}$$

More generally, let W be a complex vectorbundle on \mathcal{M}^{st} , and let $\pi^* W$ indicate its pullback under the map $\pi: \mathcal{M}_{[\gamma]}^{st} \longrightarrow \mathcal{M}^{st}$. The following equation holds.

$$\eta(W)|_{\gamma} = \frac{1}{|\text{Aut } \gamma|} i^{[\gamma]}_! \eta^{[\gamma]}(\pi^* W)$$

Proof:

Choose an embedded Kuranishi structure on $\mathcal{M}^{st}(\mathbf{B})$ for defining η , and pull back this embedded Kuranishi structure using π to define an embedded Kuranishi structure on $\mathcal{M}_{[\gamma]}^{st}(\mathbf{B})$. Let \mathcal{K} and $\pi^* \mathcal{K}$ be the associated Kuranishi categories. Take the tropical completion of these Kuranishi categories at γ , and consider the following commutative diagram.

$$\begin{array}{ccc} \mathcal{K}|_{\gamma} & \xrightarrow{ev|_{\gamma}} & (\prod_n (\text{End } \mathbf{B})^n)|_{\underline{ev}\gamma} \\ \pi \uparrow & & i^{[\gamma]} \uparrow \\ \pi^* \mathcal{K}|_{\gamma} & \xrightarrow{ev^{[\gamma]}} & (\text{End}^{[\gamma]} \mathbf{B})|_{\underline{ev}[\gamma]\gamma} \end{array}$$

When \mathcal{K} uses the family \hat{f} with automorphism group G , $\pi^* \mathcal{K}$ uses $\pi^* \hat{f}$ with the action of G induced using the universal property of $\pi^* \hat{f}$. For a given curve f with tropical part γ , there are $|\text{Aut } \gamma|$ ways of γ -decorating f so that the γ -decoration is an isomorphism. It follows that $\mathbf{F}(\pi^* \hat{f})|_{\gamma} \longrightarrow \mathbf{F}(\hat{f})|_{\gamma}$ is an $|\text{Aut } \gamma|$ -fold cover, so $\pi: \pi^* \mathcal{K}|_{\gamma} \longrightarrow \mathcal{K}|_{\gamma}$ is an $|\text{Aut } \gamma|$ -fold cover. The required formula for pushforwards follows.

□

5. ENHANCED EVALUATION MAP AND GLUING FORMULA

In this section, we enhance our evaluation map to generalize the stabilization map $ev^0: \mathcal{M}_{g,n}^{st}(\mathbf{B}) \longrightarrow \mathcal{M}_{g,n}^{st}(pt)$, where $\mathcal{M}_{g,n}^{st}(pt)$ is the moduli stack of stable exploded curves with genus g and n marked points. This stabilization map is constructed in section 4.1 of [23], where it is also shown that $\mathcal{M}_{g,n}^{st}(pt)$ is an orbifold and the

explosion of the corresponding Deligne-Mumford space relative to its boundary divisors.

For curves with genus g and n punctures where $2g - 2 + n > 0$, the target, $\mathcal{X}_{g,n}(\mathbf{B})$, of our enhanced evaluation map can be thought of as a fiber-product.

$$\begin{array}{ccc} \mathcal{X}_{g,n}(\mathbf{B}) & \longrightarrow & \mathcal{M}_{g,n}^{st}(pt) \\ \downarrow & & \downarrow \\ (\mathcal{E}\mathbf{B})^n & \longrightarrow & \cdot/\mathbf{T}^n \end{array}$$

Construct $\mathcal{X}_{g,n}(\mathbf{B})$ as the quotient stack of a groupoid $\tilde{\mathcal{X}}_{g,n}(\mathbf{B})$ with objects as follows.

$$(\tilde{\mathcal{X}}_{g,n}(\mathbf{B}))_0 = (\tilde{\mathcal{E}}_0 \mathbf{B})^n \times \mathcal{M}_{g,n+}^{st}(pt)$$

In the case $2g - 2 + n > 0$, define $\mathcal{M}_{g,n+}^{st}(pt)$ in analogy to $\mathcal{M}_{[\gamma]+}^{st}$ as the stack of stable curves in $\mathcal{M}_{g,n}^{st}$ along with an extra choice of point in each of the n ends. When (g, n) is $(0, 0)$, $(0, 1)$, or $(1, 0)$, define $\mathcal{M}_{g,n+}^{st}(pt)$ to be a point, and in the remaining case, define $\mathcal{M}_{0,2+}^{st}(pt)$ to be the stack of curves isomorphic to \mathbf{T} with an extra choice of 2 points with distinct tropical part. This $\mathcal{M}_{0,2+}^{st}(pt)$ is canonically isomorphic to $\mathbf{T}_{(0,\infty)}^1$, and to the stratum of $\mathcal{M}_{0,4}^{st}(pt)$ pairing the 1st and 3rd, and 2nd and 4th ends.

In each case, there is a (partially defined) action of \mathbf{T}^n on $\mathcal{M}_{g,n+}^{st}(pt)$, moving the n extra points. This action corresponds to the action on $\mathcal{M}_{[\gamma]+}^{st}$ in the stable case. For a curve in $\mathcal{M}_{0,2+}^{st}$, we may fix an isomorphism with \mathbf{T} so that the image of the first extra point in \mathbf{T} is before the second extra point. Then $(c_1, c_2) \in \mathbf{T}^2$ acts by multiplying the first point by c_1^{-1} and the second point by c_2 , and is defined so long as c_1/c_2 is less than the distance between the image of our points in $\mathbf{T} = \mathbb{R}$. Recalling that there is also a (partially defined) action of \mathbf{T} on $\tilde{\mathcal{E}}_0 \mathbf{B}$, let $\tilde{\mathcal{X}}_{g,n}(\mathbf{B})$ be the groupoid defined by the (partially defined) action of \mathbf{T}^n on $(\tilde{\mathcal{E}}_0 \mathbf{B})^n \times \mathcal{M}_{g,n+}^{st}(pt)$, so the morphisms in our groupoid $\tilde{\mathcal{X}}_{g,n}(\mathbf{B})$ are parametrized by the subset

$$(\tilde{\mathcal{X}}_{g,n}(\mathbf{B}))_1 \subset (\tilde{\mathcal{E}}_1 \mathbf{B})^n \times \mathcal{M}_{g,n+}^{st}(pt) \subset \mathbf{T}^n \times (\tilde{\mathcal{E}}_0 \mathbf{B})^n \times \mathcal{M}_{g,n+}^{st}(pt)$$

where the action of $c \in \mathbf{T}^n$ on $p \in (\tilde{\mathcal{E}}_0 \mathbf{B})^n \times \mathcal{M}_{g,n+}^{st}(pt)$ is defined. The two maps $(\tilde{\mathcal{X}}_{g,n}(\mathbf{B}))_1 \rightrightarrows (\tilde{\mathcal{X}}_{g,n}(\mathbf{B}))_0$ are given by $(c, p) \rightarrow p$ and $(c, p) \rightarrow c * p$. This defines the groupoid $\tilde{\mathcal{X}}_{g,n}(\mathbf{B})$. It is also convenient to take the union of these groupoids for all n and g .

$$\tilde{\mathcal{X}} := \coprod_{g,n} \tilde{\mathcal{X}}_{g,n}$$

Our evaluation map $\tilde{\mathbf{ev}}$ extends to the enhanced evaluation map

$$\tilde{EV}: \tilde{\mathcal{M}}^{st}(\mathbf{B}) \longrightarrow \tilde{\mathcal{X}}(\mathbf{B})$$

with

$$\tilde{EV}_0: \mathcal{M}_{+,+}^{st}(\mathbf{B}) \longrightarrow \tilde{\mathcal{X}}_0(\mathbf{B})$$

given by $\tilde{\mathbf{ev}}_0$ on the first factor, and the stabilization map $\mathcal{M}_{+,+}^{st}(\mathbf{B}) \longrightarrow \coprod_{g,n} \mathcal{M}_{g,n+}^{st}$ on the second factor (treating $\mathcal{M}_{g,n+}^{st}$ as a substack of $\mathcal{M}_{g,2n}^{st}$). This \tilde{EV}_0 is equivariant with respect to the various (partially defined) \mathbf{T}^n -actions, and lifts uniquely

to the map of groupoids $\tilde{E}V$ so that the following diagram commutes.

$$\begin{array}{ccc} \mathcal{M}^{st}(\mathbf{B}) & \xrightarrow{\tilde{E}V} & \tilde{\mathcal{X}}(\mathbf{B}) \\ & \searrow \mathfrak{c}\mathbf{v} & \downarrow \\ & & \coprod_n (\tilde{\mathcal{E}} \mathbf{B})^n \end{array}$$

Letting \mathcal{X} be the quotient stack of $\tilde{\mathcal{X}}$, we get the following commutative diagram of evaluation maps:

$$\begin{array}{ccc} \mathcal{M}^{st}(\mathbf{B}) & \xrightarrow{ev^0} & \coprod_{g,n} \mathcal{M}_{g,n}^{st}(pt) \\ \downarrow \mathfrak{c}\mathbf{v} & \searrow EV & \uparrow \\ \coprod_n (\mathcal{E} \mathbf{B})^n & \longleftarrow & \mathcal{X}(\mathbf{B}) \end{array}$$

In the case that \mathbf{B} is bounded and $2g - 2 + n > 0$, $\mathcal{X}_{g,n} = \mathbf{B}^n \times \mathcal{M}_{g,n}^{st}(pt)$, and EV is (the exploded version of) a familiar evaluation map used in Gromov–Witten theory.

As with $\mathfrak{c}\mathbf{v}$, we can enhance $\mathfrak{c}\mathbf{v}^{[\gamma]}$ to obtain $EV^{[\gamma]}$. For a vertex v of γ , use the notation $\mathcal{M}_{[\gamma_v]^+}^{st}(pt) := \coprod_g \mathcal{M}_{g,n^+}^{st}(pt)$, where we identify the n edges of γ_v with the n labels from $\mathcal{M}_{g,n}^{st}(pt)$. Define

$$\tilde{\mathcal{X}}_0^{[\gamma]}(\mathbf{B}) := \tilde{\mathcal{E}}_0^{[\gamma]} \mathbf{B} \times \prod_v \mathcal{M}_{[\gamma_v]^+}^{st}(pt) .$$

Then define

$$\tilde{E}V_0^{[\gamma]} : \mathcal{M}_{[\gamma]^+}^{st}(\mathbf{B}) \longrightarrow \tilde{\mathcal{X}}_0^{[\gamma]}(\mathbf{B})$$

as $\tilde{\mathfrak{c}}\mathbf{v}_0^{[\gamma]}$ on the first factor, and the cutting map followed by the stabilization map on the second factor. $\tilde{E}V_0^{[\gamma]}$ is equivariant with respect to the (partially defined) \mathbf{T} action corresponding to each edge of γ , (acting diagonally by (c, c^{-1}) on $\prod_v \mathcal{M}_{[\gamma_v]^+}^{st}(pt)$ in the case of an internal edge of γ) so we can promote $\tilde{\mathcal{X}}_0^{[\gamma]}$ to a groupoid and $\tilde{E}V_0^{[\gamma]}$ to a groupoid map so that $\tilde{\mathcal{X}}_1^{[\gamma]}(\mathbf{B})$ is a subset of $\tilde{\mathcal{E}}_1^{[\gamma]}(\mathbf{B}) \times \prod_v \mathcal{M}_{[\gamma_v]^+}^{st}(pt)$, and so that the following commutative diagram of groupoid maps exists.

$$\begin{array}{ccc} & & \tilde{\mathcal{E}}^{[\gamma]}(\mathbf{B}) \\ & \nearrow \tilde{\mathfrak{c}}\mathbf{v}^{[\gamma]} & \uparrow \\ \mathcal{M}_{[\gamma]}^{st}(\mathbf{B}) & \xrightarrow{\tilde{E}V^{[\gamma]}} & \tilde{\mathcal{X}}^{[\gamma]}(\mathbf{B}) \\ \downarrow cut & & \downarrow \tilde{\Delta} \\ \prod_v \mathcal{M}_{[\gamma_v]}^{st}(\mathbf{B}|_v) & \xrightarrow{\prod_v \tilde{E}V^{[\gamma_v]}} & \prod_v \tilde{\mathcal{X}}^{[\gamma_v]}(\mathbf{B}|_v) \end{array}$$

Passing to quotient stacks, we get the following commutative diagram

$$\begin{array}{ccc} & & \mathcal{E}^{[\gamma]}(\mathbf{B}) \\ & \nearrow \mathfrak{c}\mathbf{v}^{[\gamma]} & \uparrow \\ \mathcal{M}_{[\gamma]}^{st}(\mathbf{B}) & \xrightarrow{EV^{[\gamma]}} & \mathcal{X}^{[\gamma]}(\mathbf{B}) \\ \downarrow cut & & \downarrow \Delta \\ \prod_v \mathcal{M}_{[\gamma_v]}^{st}(\mathbf{B}|_v) & \xrightarrow{\prod_v EV^{[\gamma_v]}} & \prod_v \mathcal{X}^{[\gamma_v]}(\mathbf{B}|_v) \end{array}$$

For notational convenience, we shall use the same notation for Δ and its tropical completion.

$$\Delta: (\mathcal{X}^{[\gamma]} \mathbf{B})|_{\underline{EV}^{[\gamma]} \gamma} \longrightarrow \prod_v \mathcal{X}^{[\gamma_v]}(\mathbf{B}|_v)$$

Lemma 5.1. *The following is a fiber-product diagram.*

$$\begin{array}{ccc} (\mathcal{X}^{[\gamma]} \mathbf{B})|_{\underline{EV}^{[\gamma]} \gamma} & \longrightarrow & \mathcal{E}^{[\gamma]}(\mathbf{B})|_{\underline{ev}^{[\gamma]} \gamma} \\ \downarrow \Delta & & \downarrow \Delta \\ \prod_v \mathcal{X}^{[\gamma_v]}(\mathbf{B}|_v) & \longrightarrow & \prod_v \mathcal{E}^{[\gamma_v]}(\mathbf{B}|_v) \end{array}$$

Proof:

Applying tropical completion to the above stacks at the image of γ is equivalent to applying tropical completion to the corresponding groupoids at the image of curves with tropical part γ and extra chosen points some fixed location on γ . Although tropical completion at the image of γ_v does nothing to the stacks $\mathcal{X}^{[\gamma_v]}$ and $\mathcal{E}^{[\gamma_v]}$, applying tropical completion to the corresponding groupoids has the effect of replacing our partially defined \mathbf{T} -actions with honest \mathbf{T} -actions. After applying tropical completion at appropriate points, we get the following commutative diagram of groupoids

$$\begin{array}{ccc} \tilde{\mathcal{X}}^{[\gamma]}|_a & \longrightarrow & \tilde{\mathcal{E}}^{[\gamma]}(\mathbf{B})|_b \\ \downarrow & & \downarrow \\ \prod_v \tilde{\mathcal{X}}^{[\gamma_v]}|_c & \longrightarrow & \prod_v \tilde{\mathcal{E}}^{[\gamma_v]}(\mathbf{B}|_v)|_d \end{array}$$

where all groupoid actions are honest actions of some \mathbf{T}^n . This is a fiber product diagram, because the left side is the product of the right with the tropical completion of $\prod_v \mathcal{M}_{[\gamma_v]^+}^{st}(pt)$, at both level 0 and 1. Passing to quotient stacks therefore gives the desired fiber-product diagram of stacks.

$$\begin{array}{ccc} \mathcal{X}^{[\gamma]}|_{\underline{EV}^{[\gamma]} \gamma} & \longrightarrow & \mathcal{E}^{[\gamma]}(\mathbf{B})|_{\underline{ev}^{[\gamma]} \gamma} \\ \downarrow & & \downarrow \\ \prod_v \mathcal{X}^{[\gamma_v]} & \longrightarrow & \prod_v \mathcal{E}^{[\gamma_v]}(\mathbf{B}|_v) \end{array}$$

□

In analogy with our definition of η , define

$$\begin{aligned} \mu &:= EV_!(q^E) \\ \mu|_\gamma &:= (EV|_\gamma)_!(q^E) \\ \mu^{[\gamma]} &:= (EV^{[\gamma]}|_\gamma)_!(q^E) \end{aligned}$$

where we only need one dummy-variable, q , because genus is automatically tracked in \mathcal{X} . These pushforwards are defined as in [25] on connected components of \mathcal{X} that are orbifolds. On the other components, define μ to vanish. For W a complex vectorbundle over \mathcal{M}_\bullet^{st} , also define

$$\begin{aligned} \mu &:= EV_!(q^E c(W)) \\ \mu(W)|_\gamma &:= (EV|_\gamma)_!(q^E c(W)) \\ \mu^{[\gamma]}(W) &:= (EV^{[\gamma]}|_\gamma)_!(q^E c(W)) \end{aligned}$$

Note that $\mu^{[\gamma]}$ is a refined cohomology class on $\mathcal{X}^{[\gamma]}|_{\underline{EV}^{[\gamma]} \gamma}$, however $\mathcal{X}^{[\gamma_v]}|_{\underline{EV}^{[\gamma_v]} \gamma_v}$ coincides with $\mathcal{X}^{[\gamma_v]}$, and $\mu^{[\gamma_v]}$ is the restriction of μ to $\mathcal{X}^{[\gamma_v]} \subset \mathcal{X}(\mathbf{B}|_v)$.

Theorem 5.2.

$$\mu^{[\gamma]} = \Delta^* \prod_v \mu^{[\gamma_v]}$$

and given complex vector bundles W_v on $\mathcal{M}_{[\gamma_v]}^{st}(\mathbf{B}|_v)$,

$$\mu^{[\gamma]}(\text{cut}^* \oplus_v W_v) = \Delta^* \prod_v \mu^{[\gamma_v]}(W_v) .$$

Proof: As in the proof of Theorem 4.8, choose $ev^{[\gamma_v]}$ -submersive embedded Kuranishi structures on $\mathcal{M}_{[\gamma_v]}^{st}(\mathbf{B}|_v)$, then use Proposition 4.3 to construct a corresponding embedded Kuranishi structure on $\prod_v \mathcal{M}_{[\gamma_v]}^{st}(\mathbf{B}|_v)$ that pulls back to an embedded Kuranishi structure using cut^* . Let \mathcal{K} and $\text{cut}^* \mathcal{K}$ be the corresponding Kuranishi categories. Lemmas 5.1 and 4.4 imply that the following is a fiber-product diagram.

$$\begin{array}{ccc} \text{cut}^* \mathcal{K}|_\gamma & \xrightarrow{EV^{[\gamma]}|_\gamma} & \mathcal{X}^{[\gamma]}|_{\underline{EV}\gamma} \\ \downarrow \text{cut} & & \downarrow \Delta \\ \mathcal{K} & \xrightarrow{\prod_v EV^{[\gamma_v]}} & \prod_v \mathcal{X}^{[\gamma_v]} \end{array}$$

The only non-orbifold components of $\mathcal{X}^{[\gamma_v]}$ concern curves with genus 0, and one or two ends, all with zero multiplicity. The non-orbifold components of $\mathcal{X}^{[\gamma]}|_{\underline{EV}\gamma}$ are the inverse image of non-orbifold components of $\prod_v \mathcal{X}^{[\gamma_v]}$. In the remaining orbifold cases, Theorem 5.22 of [25] applies, and our desired formula follows from Theorems 5.22 and 6.2 of [25], as in the proof of Theorem 4.8. \square

Theorem 2.5 allows us to glue together a family of curves in $\prod_v \mathcal{M}_{[\gamma_v]}^{\infty,1}(pt)$ to obtain a family of curves in $\mathcal{M}_{[\gamma]}^{\infty,1}(pt)$. Forgetting the γ -decoration and extra points on internal edges gives a family in $\mathcal{M}_{,+}^{\infty,1}$. Composing this gluing map with the stabilization map $\mathcal{M}_{,+}^{\infty,1}(pt) \rightarrow \mathcal{M}_{,+}^{st}(pt)$, and pre-composing with the inclusion $\mathcal{M}_{[\gamma_v]}^{st}(pt) \rightarrow \mathcal{M}_{[\gamma_v]}^{\infty,1}(pt)$ gives a map,

$$\prod_v \mathcal{M}_{[\gamma_v]}^{st}(pt) \rightarrow \prod_{g,n} \mathcal{M}_{g,n}^{st}(pt)$$

compatible with the various (partially defined) actions of \mathbf{T} , and in particular, invariant under the diagonal action corresponding to each internal edge of γ . The product of this map with $\tilde{i}^{[\gamma]}: \tilde{\mathcal{E}}^{[\gamma]}(\mathbf{B}) \rightarrow (\tilde{\mathcal{E}}\mathbf{B})^n$ defines a map of groupoids

$$\tilde{I}^{[\gamma]}: \tilde{\mathcal{X}}^{[\gamma]} \rightarrow \tilde{\mathcal{X}}$$

compatible with our evaluation maps. Passing to quotient stacks and taking tropical completion at the image of γ gives a map

$$I^{[\gamma]}: \mathcal{X}^{[\gamma]}|_{\underline{EV}\gamma} \rightarrow \mathcal{X}|_{\underline{EV}\gamma}$$

Lemma 5.3. *The following relationship holds between μ and $\mu^{[\gamma]}$*

$$\mu|_\gamma = \frac{1}{|\text{Aut } \gamma|} I^{[\gamma]}_* \mu^{[\gamma]}$$

(where $I^{[\gamma]}_* \mu^{[\gamma]}$ is defined to be zero on non-orbifold components of $\mathcal{X}|_\gamma$). More generally, let W be a complex vectorbundle on $\mathcal{M}_{,+}^{st}$ and let $\pi^* W$ indicate its pullback

under the map $\pi: \mathcal{M}_{[\gamma]}^{st} \rightarrow \mathcal{M}^{st}$. Then

$$\mu(W)|_{\gamma} = \frac{1}{|\text{Aut } \gamma|} I_!^{[\gamma]} \mu^{[\gamma]}(\pi^* W)$$

Proof:

Choose embedded Kuranishi structures and use notation as in the proof of Lemma 4.9. In this case, we must consider the following commutative diagram.

$$\begin{array}{ccc} \mathcal{K}|_{\gamma} & \xrightarrow{EV|_{\gamma}} & \mathcal{X}|_{\underline{EV}\gamma} \\ \uparrow \pi & & \uparrow I^{[\gamma]} \\ \pi^* \mathcal{K}|_{\gamma} & \xrightarrow{EV^{[\gamma]}|_{\gamma}} & \mathcal{X}^{[\gamma]}|_{\underline{EV}^{[\gamma]}\gamma} \end{array}$$

As in the proof of Lemma 4.9, π is an $|\text{Aut } \gamma|$ -fold cover. If all components of the righthand side of the above diagram were orbifolds, our formula would follow immediately. We need to check that components of $\pi^* \mathcal{K}|_{\gamma}$ sent to non-orbifold components of $\mathcal{X}^{[\gamma]}|_{\underline{EV}^{[\gamma]}\gamma}$ do not contribute anything to $\mu(W)|_{\gamma}$. If a curve with tropical part γ is sent to a non-orbifold component of $\mathcal{X}^{[\gamma]}$, some vertex v of γ must have one or two edges, all with 0 multiplicity, and the stratum of our curve labelled by v must have genus 0. Within $\pi^* \mathcal{K}$, such a curve must be contained in a family of curves that allows all possible lengths for the corresponding edges. In each case, there is a $\mathbf{T}_{(0,\infty)}^1$ -worth of choice for these edge lengths that is crunched to a single point under $I^{[\gamma]} \circ EV^{[\gamma]}|_{\gamma}$, therefore the corresponding connected component of $\mathcal{K}|_{\gamma}$ does not contribute to $\mu(W)$.

As all connected components of $\pi^* \mathcal{K}|_{\gamma}$ that contribute to $\mu(W)$ are sent to orbifold components of $\mathcal{X}^{[\gamma]}|_{\underline{EV}^{[\gamma]}\gamma}$, we may use the formula $(EV|_{\gamma} \circ \pi)_! = I_!^{[\gamma]} \circ (EV^{[\gamma]}|_{\gamma})_!$, and the fact that π is an $|\text{Aut } \gamma|$ -fold cover to prove the desired relationship.

$$\mu(W)|_{\gamma} = \frac{1}{|\text{Aut } \gamma|} I_!^{[\gamma]} \mu^{[\gamma]}(\pi^* W)$$

□

For curves in a smooth symplectic manifold B , Lemma 5.3 and Theorem 5.2 recover Kontsevich and Mannin's splitting and genus-reduction axioms of Gromov-Witten invariants. For example, the genus-reduction axiom may be understood as follows: The tropical part of B is a single point. The tropical part of an exploded curve corresponding to a curve in B with a single, non-separating node, genus g and n punctures is a tropical curve γ with a single vertex, n ends corresponding to punctures and a single interior edge corresponding to the node. After fixing the labeling of the n ends, there are 2 automorphisms of γ , the nontrivial one reversing the interior edge.

The relevant component of $\mathcal{X}(B)$ is $\mathcal{X}_{g,n}(B) = B^n \times \mathcal{M}_{g,n}$, where $\mathcal{M}_{g,n}$ is the moduli stack of stable exploded curves with genus g and n labeled ends. This may also be thought of as the explosion of the corresponding Delign-Mumford stack, $\bar{\mathcal{M}}_{g,n}$; see section 4.1 of [23]. Similarly, the relevant component of $\mathcal{X}^{[\gamma_v]}(B|_v)$ is $B^{n+2} \times \mathcal{M}_{g-1,n+2}$, and the genus g component of $\mathcal{X}^{[\gamma]}$ is a $\mathbf{T}_{(0,\infty)}^1$ -bundle over $B^{n+1} \times \mathcal{M}_{g-1,n+2}$. This extra $\mathbf{T}_{(0,\infty)}^1$ bundle is created as follows. Take the $(\mathbf{T}_{(0,\infty)}^1)^2$ -bundle over $\mathcal{M}_{g-1,n+2}$ given by choosing an extra point in each of the last two edges, and quotient this bundle by the (partially defined) diagonal action of \mathbf{T} multiplying by (c, c^{-1}) . This $\mathbf{T}_{(0,\infty)}^1$ -bundle over $\mathcal{M}_{g-1,n+2}$ is also a \mathbb{Z}_2 -fold

cover of the stratum of $\mathcal{M}_{g,n}$ corresponding to curves with a single vertex and single internal edge. The \mathbb{Z}_2 -action swaps the labels of the last two edges in $\mathcal{M}_{g-1,n+2}$. In terms of Deligne-Mumford space, the \mathbb{Z}_2 -quotient of this $\mathbf{T}_{(0,\infty)}^1$ -bundle corresponds to the \mathbb{C}^* -bundle obtained by removing the zero-section of the normal bundle of $\bar{M}_{g-1,n+2}/\mathbb{Z}_2 \subset \bar{M}_{g,n}$. When we apply tropical completion, we replace this $\mathbf{T}_{(0,\infty)}^1$ -bundle with a \mathbf{T} -bundle, $\mathbf{T} \rtimes \mathcal{M}_{g-1,n+2}$.

Our gluing formula is

$$\mu|_{\gamma} = \frac{1}{2} I!^{[\gamma]} \Delta^* \mu^{[\gamma_v]}$$

stated in terms of the maps

$$B^n \times B^2 \times \mathcal{M}_{g-1,n+2} \xleftarrow{\Delta} B^n \times B \times \mathbf{T} \rtimes \mathcal{M}_{g-1,n+2} \xrightarrow{I^{[\gamma]}} B^n \times (\mathbf{T} \rtimes \mathcal{M}_{g-1,n+2})/\mathbb{Z}_2$$

where Δ is the product of the identity on B^n with the diagonal $B \rightarrow B^2$ and the bundle map $\mathbf{T} \rtimes \mathcal{M}_{g-1,n+2} \rightarrow \mathcal{M}_{g-1,n+2}$, and $I^{[\gamma]}$ is the composition of a projection, π , crushing the extra factor of \mathbf{B} , and a \mathbb{Z}_2 -fold covering map,

$$\psi: \mathbf{B}^n \times (\mathbf{T} \rtimes \mathcal{M}_{g-1,n+2}) \rightarrow \mathbf{B}^n \times (\mathbf{T} \rtimes \mathcal{M}_{g-1,n+2})/\mathbb{Z}_2.$$

Our gluing formula may be rewritten as follows.

$$(15) \quad \psi^* \mu|_{\gamma} = \pi! \Delta^* \mu^{[\gamma_v]}$$

Taking smooth parts of our maps above gives the following diagram.

$$\begin{array}{ccc} \mathcal{M}_{g,n}^{st}(B) & \xrightarrow{\quad\quad\quad} & B^n \times \bar{M}_{g-1,n+2}/\mathbb{Z}_2 \subset B^n \times \bar{M}_{g,n} \\ & & \uparrow [\psi] \\ B^{n+1} \times \bar{M}_{g-1,n+2} & \xrightarrow{[\pi]} & B^n \times \bar{M}_{g-1,n+2} \\ & \downarrow [\Delta] & \\ \mathcal{M}_{g-1,n+2}^{st}(B) & \xrightarrow{\quad\quad\quad} & B^{n+2} \times \bar{M}_{g-1,n+2} \end{array}$$

Let $C_{g-1,n+2}$ be the pushforward of $\mu^{[\gamma_v]}$ in $H^*(B^{n+2} \times \bar{M}_{g-1,n+2})$ using the smooth part map, $\mathcal{M}_{g-1,n+2} \rightarrow \bar{M}_{g-1,n+2}$, and let $C_{g,n}$ in $H^*(B^n \times \bar{M}_{g,n})$ be the pushforward of μ . Konsevich and Mannin's genus-reduction formula from [8] can be restated²⁸ as

$$[\psi]^* C_{g,n} = [\pi]! [\Delta]^* C_{g-1,n+2}$$

which is implied by our gluing formula, (15), because each stage of pushing forward or pulling back commutes with pushing forward using the smooth part map.

The splitting axiom is proved similarly, except now γ is a tropical curve with 2 vertices connected to one internal edge, and n_i external edges attached to the i th vertex. Now γ has no symmetries (assuming $n \neq 0$). The relevant commutative diagram of maps for genus g invariants is below. (The disjoint unions below are

²⁸Actually, the genus-reduction axiom from [8] also keeps track of the homology class of curves, whereas our formula only keeps track of their ω -energy. In this case, our formula can easily be upgraded to keep track of homology classes as outlined in section 6.

over choices of nonnegative integers g_i so that $g_1 + g_2 = g$.)

$$\begin{array}{ccc}
B^n \times \mathcal{M}_{g,n} & \xrightarrow{[\cdot]} & B^n \times \bar{\mathcal{M}}_{g,n} \\
& & \uparrow \psi \\
B^n \times \mathcal{M}_{g,n}|_{\check{\gamma}} & \xrightarrow{[\cdot]} & B^n \times \coprod \bar{\mathcal{M}}_{g_1,n_1+1} \times \bar{\mathcal{M}}_{g_2,n_2+1} \\
\uparrow I^{[\gamma]} & & \uparrow \pi \\
B^{n+1} \times \mathbf{T} \times \coprod \mathcal{M}_{g_1,n_1+1} \times \mathcal{M}_{g_2,n_2+1} & \xrightarrow{[\cdot]} & B^{n+1} \times \coprod \bar{\mathcal{M}}_{g_1,n_1+1} \times \bar{\mathcal{M}}_{g_2,n_2+1} \\
\downarrow \Delta & & \downarrow [\Delta] \\
B^{n+2} \times \coprod \mathcal{M}_{g_1,n_1+1} \times \mathcal{M}_{g_2,n_2+1} & \xrightarrow{[\cdot]} & B^{n+2} \times \coprod \bar{\mathcal{M}}_{g_1,n_1+1} \times \bar{\mathcal{M}}_{g_2,n_2+1}
\end{array}$$

Our gluing formula, $\mu|_{\check{\gamma}} = I^{[\gamma]}_! \Delta^*(\mu^{[\gamma_{v_1}]} \wedge \mu^{[\gamma_{v_2}]})$, implies

$$\psi^* C_{g,n} = \sum_{g_1+g_2=g} \pi_! [\Delta]^*(C_{g_1,n_1} \wedge C_{g_2,n_2})$$

which is the splitting axiom from [8].

6. FURTHER GLUING FORMULAE

We now have two gluing formulae. The first is

$$\eta|_{\check{\gamma}} = \frac{k_{\gamma}}{|\text{Aut } \gamma|} i^{[\gamma]}_! \Delta^* \prod_v \eta^{[\gamma_v]}$$

where k_{γ} is the product of the multiplicities of the internal edges of γ , and η and $\eta^{[\gamma_v]}$ are the pushforward of $q^E \hbar^{2g-2+n}$ via the maps

$$ev: \mathcal{M}^{st} \mathbf{B} \longrightarrow \coprod_n (\text{End } \mathbf{B})^n$$

$$ev^{[\gamma_v]}: \mathcal{M}^{st}_{[\gamma_v]}(\mathbf{B}|_v) \longrightarrow \text{End}^{[\gamma_v]} \mathbf{B}$$

and Δ and $i^{[\gamma]}$ are natural maps

$$\prod_v \text{End}^{[\gamma_v]}(\mathbf{B}|_v) \xleftarrow{\Delta} \text{End}^{[\gamma]}(\mathbf{B})|_{\underline{ev}^{[\gamma]}\gamma} \xrightarrow{i^{[\gamma]}} (\prod_n (\text{End } \mathbf{B})^n)|_{\underline{ev}\gamma}.$$

Our second gluing formula

$$\mu|_{\check{\gamma}} = \frac{1}{|\text{Aut } \gamma|} I^{[\gamma]}_! \Delta^* \prod_v \mu^{[\gamma_v]}$$

involves the pushforward, μ and $\mu^{[\gamma_v]}$, of q^E using enhanced evaluation maps

$$EV: \mathcal{M}^{st} \mathbf{B} \longrightarrow \mathcal{X} \mathbf{B}$$

$$EV^{[\gamma_v]}: \mathcal{M}^{st}_{[\gamma_v]}(\mathbf{B}|_v) \longrightarrow \mathcal{X}^{[\gamma_v]}(\mathbf{B}|_v)$$

and the natural maps

$$\prod_v \mathcal{X}^{[\gamma_v]}(\mathbf{B}|_v) \xleftarrow{\Delta} (\mathcal{X}^{[\gamma]} \mathbf{B})|_{\underline{EV}^{[\gamma]}\gamma} \xrightarrow{I^{[\gamma]}} (\mathcal{X} \mathbf{B})|_{\underline{EV}\gamma}.$$

Our first gluing formula follows from Theorem 4.8 and Lemma 4.9, and the second from Theorem 5.2 and Lemma 5.3. These two key theorems follow from

the fact that the left two squares in the following diagram become fiber-product diagrams after applying tropical completion suitably, as in Lemmas 4.4 and 5.3.

$$\begin{array}{ccccccc}
 & & & \xrightarrow{ev^{[\gamma]}} & & & \\
 & & \xrightarrow{\epsilon v^{[\gamma]}} & & \xrightarrow{\epsilon v^{[\gamma]}} & & \\
 \mathcal{M}_{[\gamma]}^{st}(\mathbf{B}) & \xrightarrow{EV^{[\gamma]}} & \mathcal{X}^{[\gamma]} & \longrightarrow & \mathcal{E}^{[\gamma]} \mathbf{B} & \longrightarrow & \text{End}^{[\gamma]} \mathbf{B} \\
 \downarrow \text{cut} & & \downarrow \Delta & & \downarrow \Delta & & \downarrow \Delta \\
 \prod_v \mathcal{M}_{[\gamma_v]}^{st}(\mathbf{B}|_v) & \xrightarrow{\prod_v EV^{[\gamma_v]}} & \prod_v \mathcal{X}^{[\gamma_v]}(\mathbf{B}|_v) & \longrightarrow & \prod_v \mathcal{E}^{[\gamma_v]}(\mathbf{B}|_v) & \longrightarrow & \prod_v \text{End}^{[\gamma_v]}(\mathbf{B}|_v) \\
 & & \xrightarrow{\prod_v \epsilon v^{[\gamma_v]}} & & \xrightarrow{\prod_v \epsilon v^{[\gamma_v]}} & & \\
 & & & \xrightarrow{\prod_v ev^{[\gamma_v]}} & & &
 \end{array}$$

We can also construct gluing formulae keeping track of more discrete information. Let $\hat{\mathcal{X}}$ and $\hat{\mathcal{X}}^{[\gamma_v]}$ be covers of \mathcal{X} and $\mathcal{X}^{[\gamma_v]}$ respectively with lifts \hat{EV} and $\hat{EV}^{[\gamma_v]}$ of our evaluation maps.

$$\begin{array}{ccc}
 & \hat{\mathcal{X}} & \\
 \hat{EV} \nearrow & & \downarrow \\
 \mathcal{M}^{st}(\mathbf{B}) & \xrightarrow{EV} & \mathcal{X}
 \end{array}
 \qquad
 \begin{array}{ccc}
 & \hat{\mathcal{X}}^{[\gamma_v]} & \\
 \hat{EV}^{[\gamma_v]} \nearrow & & \downarrow \\
 \mathcal{M}_{[\gamma_v]}^{st}(\mathbf{B}|_v) & \xrightarrow{EV} & \mathcal{X}^{[\gamma_v]}
 \end{array}$$

Suppose further that there is a lift, $\hat{I}^{[\gamma]}$ of $I^{[\gamma]}$ as in the diagram below, compatible with \hat{EV} and $\hat{EV}^{[\gamma_v]}$.

$$\begin{array}{ccccc}
 \prod_v \hat{\mathcal{X}}^{[\gamma_v]} & \xleftarrow{\Delta} & (\prod_v \hat{\mathcal{X}}^{[\gamma_v]}) \times_{\prod_v \mathcal{X}^{[\gamma_v]}} \mathcal{X}^{[\gamma]}|_{\underline{EV}\gamma} & \xrightarrow{\hat{I}^{[\gamma]}} & \hat{\mathcal{X}}|_{\underline{EV}\gamma} \\
 \downarrow & & \downarrow & & \downarrow \\
 \prod_v \mathcal{X}^{[\gamma_v]} & \xleftarrow{\Delta} & \mathcal{X}^{[\gamma]}|_{\underline{EV}\gamma} & \xrightarrow{I^{[\gamma]}} & \mathcal{X}|_{\underline{EV}\gamma}
 \end{array}$$

The following gluing formula then holds,

$$\hat{\mu}|_{\gamma} = \frac{1}{|\text{Aut } \gamma|} \hat{I}^{[\gamma]}_{!} \Delta^* \prod_v \hat{\mu}^{[\gamma_v]}$$

where

$$\hat{\mu} := \hat{EV}_!(q^E) \text{ and } \hat{\mu}^{[\gamma_v]} := \hat{EV}_!^{[\gamma_v]}(q^E) .$$

We can also include the contribution of gravitational descendants, as in Theorem 5.2 and Lemma 5.3.

An example of such a lift is as follows: we can lift EV to keep track of the integral over curves of all closed 2-forms α in $\Omega^2(\mathbf{B})$, and lift $EV^{[\gamma_v]}$ to record the integral of $\alpha|_v$ over curves.

Of course, we can similarly extend our first gluing formula by suitably lifting ev . An example of this construction is the use of rim tori in [6], explained further in [27].

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E-mail address: brettdeparker@gmail.com